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A Stability-Rhobustness Margin in Discrete Time

by

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Excerpt from a letter to Prof. Wilson J. Rugh of the Dept. of Electrical & Computer Engineering at the Johns Hopkins University dated August 13, 2002 in which I told him about my having generalized from analog time to discrete-time (in connection with an evening course on DSP for the local branch of Florida Inst. of Technology that I was then teaching) my 'Rhobustness' Criterion that I had first published for analog-time systems in 1956 [**1**] and then improved in an IEEE Conference paper [**2**] in 1991 after use in various applications [**3**]-[**5**]):

I have just derived what I believe is an elegant generalization from continuous-time systems to discrete-time systems of my theory of "rhobustification" of control and observer systems, as follows:

In your book [**6,** *pp*. 406-412] you prove that if the system

$$x^{k+1} = A\cdot x^k + B\cdot\cdot u^k, \quad u = -K\cdot x,$$

is stabilized by state-feedback, to become [closed-loop]

$$x^{k+1} = A_{cl}\cdot x^k, \qquad x^0 = x^o, \qquad A_{cl} \equiv: A - B\cdot K,$$

then there must be positive numbers $\gamma \geq 1$ and $\lambda < 1$ [$\lambda \equiv: \max\{|\mathrm{eig}(A_{cl})|\}$] such that

$$\|x^k\| \leq \gamma\cdot\|x^o\|\cdot\lambda^k, \qquad (k = 0, 1, 2, 3, \ldots).$$

I now define a *stability "rhobustness" margin* ρ, ($0 < \rho < 1$), by $\rho \equiv: (1-\lambda)/\gamma$, and prove that if in ***NONSTATIONARY/NONLINEAR & EXTERNALLY-FORCED*** actuality the system is better-modeled by the 'perturbed' system

$$x^{k+1} = A_{cl}\cdot x^k + f(k, x^k) + g(k), \qquad (k = 0, 1, 2, 3, \ldots),$$

where there exist (κ,δ) such that, for all x and k, $\|f(k, x)\| \leq \kappa\cdot\|x\|$, $\|g(k)\| \leq \delta$, and where $\kappa < \rho$, then

$$\|x^k\| \leq \gamma\cdot\|x^o\|\cdot\Lambda^k + [\delta/(\rho-\kappa)]\cdot(1-\Lambda^k), \quad \Lambda \equiv \{\lambda + [\kappa/\rho]\cdot(1-\lambda)\} < 1, \qquad (k = 0, 1, 2, 3, \ldots).$$

Hence the ***larger*** is ρ, the ***LESS SENSITIVE*** is the actual system to *UNMODELLED* **cross-couplings** & other *NONSTATIONARY* and/or *NONLINEAR* effects, as well as *EXTERNAL FORCING!*

In addition to the STABILITY rhobustness margin $\rho \equiv: (1-\lambda)/\gamma$, one may also define a FIDELITY robustness factor or *stochastic-disturbance rejection factor* $\boldsymbol{\sigma} \equiv: (1-\lambda^2)^{1/2}/\gamma$, which has the property that if one adds to the right-hand-side of the above system a zero-mean stochastic process of non-negative-definitive symmetric covariance matrix $Q = Q^T \geq 0$ then the system's state becomes a sequential stochastic process with the above-given sequence $\{\hat{x}^k\}$ as its mean values and a zero-mean dispersion $\tilde{x}^k = x^k - \hat{x}^k$ whose covariance matrix $P = P^T > 0$ satisfies $\|P\| \leq \|Q\|/\boldsymbol{\sigma}^2$, so that *e.g.* if $Q = \sigma_d^2 \cdot I_n$ is, as is often assumed, a multiple of the n-dimensional identity matrix I_n, then $\|\tilde{x}^k\| \leq (\sigma_d/\boldsymbol{\sigma})$.

When design-parameters (*e.g.* the feedback gain matrix K above) are varied, the maximum values of $\rho \equiv: (1-\lambda)/\gamma$ and $\boldsymbol{\sigma} \equiv: (1-\lambda^2)^{1/2}/\gamma$ are usually attained separately, so that one must decide on the desired *tradeoff* between stability rhobustness & fidelity robustness in order to make an "optimal" design. Fortunately, it is usually possible to choose a design in which both criteria are near to their maxima, e.g. by taking $\boldsymbol{\xi} = (\rho + \boldsymbol{\sigma})/2$ as the criterion to be maximized.

The derivation of the preceding results is not trivial, so that we now turn to a more formal presentation. In the sequel, by $\|x\| \equiv: (x^T \cdot x)^{1/2} \geq 0$ we denote the Euclidean length of a real n-vector x, where ^{T} denotes transposition, and, also, $\|M\| \equiv: \max\{\|M \cdot x\|/\|x\| \mid \forall x \in \Re^n, \|x\| > 0\}$.

THEOREM. Consider a Linear Time-Invariant (LTI) state-vector dynamical system in discrete time for which the origin $x = 0$ of real Euclidean n-space $\Re^n$ is an exponentially-asymptotically *stable equilibrium*, i.e.

(1a) $$x^{k+1} = A \cdot x^k, \quad x^0 = x^o, \quad (k = 0, 1, 2, 3, \ldots),$$

a necessary & sufficient condition for which is that all of the poles of the *resolvent* $(z \cdot I_n - A)^{-1}$ of A, namely the roots of its *characteristic polynomial* $\Delta(z) \equiv: det(z \cdot I_n - A) = 0$, should lie inside the unit circle of the complex z-plane $\mathscr{C}^2$. Then either by expanding the resolvent in partial fractions by means of the Calculus of Residues, or by making use of the Jordan canonical form, it is possible to compute positive numbers $\gamma = \gamma(A) \geq 1$ and $0 < \lambda = \lambda(A) < 1$ such that

(2a) $$\|A^k\| \leq \gamma \cdot \lambda^k, \quad (k = 0, 1, 2, 3, \ldots),$$

as a result of which, together with the fact that the general solution of (1a) is $x^k = A^k \cdot x^o$,

(2b) $$\|x^k\| \leq \gamma \cdot \|x^o\| \cdot \lambda^k, \quad (k = 0, 1, 2, 3, \ldots).$$

Now suppose that (1a) is an idealized model of the true or perturbed system

(1b) $$x^{k+1} = A \cdot x^k + f(k, x^k) + g(k), \quad (k = 0, 1, 2, 3, \ldots),$$

where there exist positive numbers (κ, δ) such that, for all x and k,

(1c) $$\|f(k, x)\| \leq \kappa \cdot \|x\|, \quad \|g(k)\| \leq \delta,$$

and where $\kappa < \rho$, with $0 < \rho < 1$ defined by

(1d) $$\rho \equiv: (1-\lambda)/\gamma,$$

Then there is a spherical neighborhood of the origin, defined by

$$\|x\| \;\le\; \delta/(\rho - \kappa), \tag{1e}$$

which is a global asymptotically stable attractor in the sense that every solution of (1b) must satisfy

$$\|x^k\| \;\le\; \gamma\cdot\|x^o\|\cdot\Lambda^k \;+\; [\delta/(\rho-\kappa)]\cdot(1-\Lambda^k)\,, \qquad (k \;=\; 0, 1, 2, 3, ...), \tag{3}$$

where

$$0 \;<\; \Lambda \;\equiv:\; \{\, \lambda + [\kappa/\rho]\cdot(1-\lambda)\} \;<\; 1. \tag{4}$$

PROOF. It is well-known [**6**], and readily proved by induction, that the discrete-time version of Lagrange's "Variation of Constants Formula" applies to provide the general solution of (1b), for (k = 0, 1, 2, 3, ...), in the form

$$x^k \;=\; A^k\cdot x^o \;+\; \sum_{j=0}^{k-1} A^{k-1-j}\cdot[f(j, x^j) \;+\; g(j)], \tag{5}$$

whence, making use of (2a), it is easy to see [after multiplication by λ^{-k}] that if we define φ by

$$\varphi(k) \;\equiv:\; \gamma\cdot\|x^o\| + (\kappa\cdot\gamma/\lambda)\cdot\sum_{j=0}^{k-1} \lambda^{-j}\cdot\|x^j\| \;+ (\delta\cdot\gamma/\lambda)\cdot\sum_{j=0}^{k-1} \lambda^{-j}, \tag{6}$$

then

$$\lambda^{-k}\cdot\|x^k\| \;\le\; \varphi(k), \tag{7}$$

and so, from (6) & (7)

$$\varphi(k+1) - \varphi(k) \;=\; (\kappa\cdot\gamma/\lambda)\cdot\lambda^{-k}\cdot\|x^k\| \;+\; (\delta\cdot\gamma/\lambda)\cdot\lambda^{-k} \;\le \tag{8a}$$

$$\le\; (\kappa\cdot\gamma/\lambda)\cdot\varphi(k) \;+ (\delta\cdot\gamma/\lambda)\cdot\lambda^{-k}\,. \tag{8b}$$

Now define

$$\psi(k) \;\equiv:\; \lambda^k\cdot\varphi(k), \tag{9a}$$

$$\varphi(k) \;\equiv\; \lambda^{-k}\cdot\psi(k), \tag{9b}$$

and find [after canceling two equal & opposite terms] that

$$\varphi(k+1) - \varphi(k) \;\equiv\; [(1-\lambda)/\lambda]\cdot\lambda^{-k}\cdot\psi(k+1) \;+\; \lambda^{-k}\cdot[\psi(k+1) - \psi(k)]. \tag{10}$$

Next, substitute (10) into (8a) & (8b) and simplify via (9b) to get

$$[(1-\lambda)/\lambda]\cdot\lambda^{-k}\cdot\psi(k+1) \;+\; \lambda^{-k}\cdot[\psi(k+1) - \psi(k)] \;\le\; (\kappa\cdot\gamma/\lambda)\cdot\lambda^{-k}\cdot\psi(k) \;+ (\delta\cdot\gamma/\lambda)\cdot\lambda^{-k}, \tag{11}$$

whence, upon multiplying through by λ^k, we get, after further simplification by cancellation,

$$(1/\lambda)\cdot\psi(k+1) \;-\; \psi(k) \;\le\; (\kappa\cdot\gamma/\lambda)\cdot\psi(k) \;+ (\delta\cdot\gamma/\lambda)\,, \tag{12}$$

or

(13) $$\psi(k+1) \le (\lambda + \kappa\cdot\gamma)\cdot\psi(k) + \delta\cdot\gamma.$$

Now note that, because $\kappa < \rho = (1 - \lambda)/\gamma$,

(14) $$\lambda + \kappa\cdot\gamma < \lambda + \rho\cdot\gamma = \lambda + [(1 - \lambda)/\gamma]\cdot\gamma = \lambda + (1 - \lambda) = 1,$$

so we may define

(15) $$\Lambda \equiv: \lambda + \kappa\cdot\gamma , \quad 0 < \Lambda < 1,$$

and express (13) as

(16) $$\psi(k+1) \le \Lambda\cdot\psi(k) + \delta\cdot\gamma,$$

which [because by (6), $\psi(0) = \varphi(0) = \gamma\cdot\|x^o\|$], may be solved as usual by induction and use of the finite form of the geometric series to yield

(17) $$\psi(k) \le \Lambda^k\cdot\psi(0) + \{\Lambda^{k-1} + \Lambda^{k-2} + \cdots + \Lambda + 1\}\cdot\delta\cdot\gamma =$$
$$= \gamma\cdot\|x^o\|\cdot\Lambda^k + [(1 - \Lambda^k)/(1 - \Lambda)]\cdot\delta\cdot\gamma =$$
$$= \gamma\cdot\|x^o\|\cdot\Lambda^k + \Theta\cdot\delta\cdot(1 - \Lambda^k),$$

where, using $\gamma = (1 - \lambda)/\rho$,

(18) $$\Theta \equiv: [(1 - \lambda)/(1 - \Lambda)]/\rho = (1 - \lambda)/\{[1 - (\lambda + (\kappa/\rho)(1 - \lambda)]\cdot\rho\} =$$
$$= (1 - \lambda)/\{[(1 - \lambda) - (\kappa/\rho)(1 - \lambda)]\cdot\rho\} = 1/(\rho - \kappa),$$

which, upon inserting (18) into (17), yields

(19) $$\psi(k) \le \gamma\cdot\|x^o\|\cdot\Lambda^k + (1 - \Lambda^k)\cdot[\delta/(\rho - \kappa)].$$

Now, by (7) and (9b),

(20) $$\lambda^{-k}\cdot\|x^k\| \le \varphi(k) \equiv \lambda^{-k}\cdot\psi(k),$$

whence, finally, canceling λ^{-k} and using (19),

(21) $$\|x^k\| \le \gamma\cdot\|x^o\|\cdot\Lambda^k + [\delta/(\rho - \kappa)]\cdot(1 - \Lambda^k),$$

which proves (3) and thus completes the proof of the theorem.

COROLLARY. Let the perturbed system be now

(1f) $$x^{k+1} = A\cdot x^k + h(k), \qquad (k = 0, 1, 2, 3, ...),$$

where, using $\mathcal{E}$ to denote the statistical Expectation operator, $\mathcal{E}\{h(k)\} = 0$, and assume also that

(1g) $$\mathcal{E}\{h(j)\cdot h(k)^T\} = Q\cdot\delta_{jk} , \quad Q = Q^T \ge 0,$$

where δ_{jk} denotes the Kronecker delta-function. Then, setting $\hat{x}^k = \mathcal{E}\{x^k\}$ and $\tilde{x}^k = x^k - \hat{x}^k$, we may define $P_k = (P_k)^T > 0$ by $P_k = \mathcal{E}\{\tilde{x}^k(\tilde{x}^k)^T\}$ and use (5) to get

$$\text{(22)} \qquad \tilde{x}^k = \sum_{j=0}^{k-1} A^{k-1-j}\cdot h(j),$$

whence, as usual

$$\text{(23)} \qquad P_k = \sum_{j=0}^{k-1} A^{k-1-j}\cdot Q\cdot(A^{k-1-j})^T,$$

from which, as in (17) above,

$$\text{(24)} \qquad \|P_k\| \leq \gamma^2\cdot\|Q\|\cdot\{(\lambda^2)^{k-1} + (\lambda^2)^{k-2} + \cdots + \lambda^2 + 1\} \leq \gamma^2\cdot\|Q\|/(1-\lambda^2) = \|Q\|/\boldsymbol{\sigma}^2,$$

where the *dispersion-reduction factor* $\boldsymbol{\sigma}$ is defined as

$$\text{(25)} \qquad \boldsymbol{\sigma} \equiv: (1-\lambda^2)^{1/2}/\gamma\ .$$

Note that because $0 < \lambda < 1$, it is always the case that $(1-\lambda)^2 \leq (1-\lambda^2)$, whence in general

$$\text{(26)} \qquad \rho \leq \boldsymbol{\sigma},$$

although, as already mentioned, it is usually possible to "nearly maximize" both stability rhobustness ρ and fidelity robustness $\boldsymbol{\sigma}$ simultaneously..

References

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