Robust Controller Design for the GIIB Electrostatic Gyro via Rho Synthesis†

by

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ABSTRACT

The Electro-Static Gyro (ESG) provides an inertial reference for the Trident II navigation system Inertial Measurement Unit (IMU) platform stabilization function. Years of laboratory and field data show the presence of time and latitude dependent limit-cycles and variable offsets. Current work includes an improved analysis model for prediction of nonlinear dynamical effects and synthesis of an alternative gyro controller for both comparison with the existing configuration and possible future use. The ESG's solid Beryllium rotor, 1 cm in diameter, is very accurately suspended at the center of a spherical capacitance actuation chamber. This actuation cavity is divided into four diametrically opposed plate pairs, providing non-orthogonal actuator channels. The four-axis controller applies plate-to-rotor voltage proportional to centering error along each axis, but axis redundancy creates inherently conflicting suspension forces on the rotor which require electronic compensation to harmonize the control actuation signals. The existing controller was originally designed by Walter Evans using his Root Locus methodology on a sequence of Single-Input Single-Output (SISO) approximations and multivariable/nonlinear simulations for design finalization.

This paper presents a preliminary design of a gyro rotor controller based upon the Multiple-Input Multiple Output (MIMO) Rho Synthesis† procedure. A standard state-space linearized model $\dot{x} = Ax + Bu$, $y = Cx$ is augmented with *uncertain* internal nonlinear perturbations $\Delta A(t,x)\cdot x$ and *uncertain* external deviations $B\cdot g(t)$ to represent known nonlinearities (such as from dead-zones in the actuators) and time-dependencies (such as from latitude). The external perturbations are essentially eliminated by "synthetic feedforward" using C.D. Johnson's Disturbance Accommodating Control (DAC) methodology (which systematically unifies and combines the known but *ad hoc* virtues of integral feedback, notch filters, load feedforward, etc.), thereby providing an on-line robust estimate $\hat{g}$ of g which is nearly eliminated by replacing the LQG regulator-feedback law $u = -K\hat{x}$ by $u = -K\hat{x} - \hat{g}$. Internal perturbations are accommodated by designing the "rhobustness margin" $\rho = \rho(A_{cl})$ of the ideal closed-loop dynamical coefficient matrix $A_{cl} = A - BK$ to be sufficient that $\|\Delta A(t,x)\| \le \kappa < \rho$, which ensures globally uniform exponential asymptotic stability of the equilibrium state $x = 0$ while permitting no increase of the overshoot factor and only marginal increase of response time.

More detailed nonlinear simulation of the system under ρ-control shows the elimination of two known limit-cycles while maintaining excellent dynamic response and producing enhanced disturbance rejection capabilities.

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†Rho-synthesis is a public-domain Computer Aided Control Engineering (CACE) procedure based upon prior publications of R.W. Bass and available (in a *MATLAB* implementation) *gratis* to registered *MATLAB* users as an add-on *MATLAB* Toolkit upon written request from R.W. Bass.

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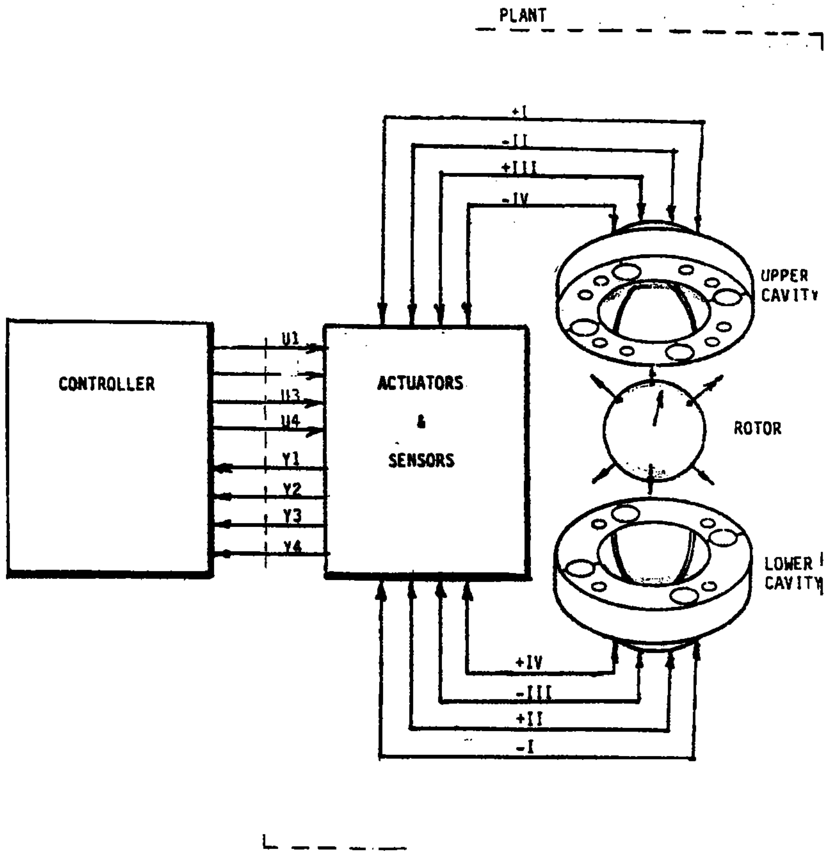
G11B Gyro – Pictorial Block Diagram



Control Objectives

(1) Rotor Centering <= Main Focus
(2) Rotor Angular Velocity <= Later

Disturbances

(1) Shock <== Main focus of this paper
Vd=(.36e6)texp(−30t)sin2pi66.67t
(2) Vibration <== Later
Vd= 98cos2pi1000t
(3) Limit Cycles <== eliminated via Rho

Parameter Uncertainty

Plant 1%

Actuators/Sensors 5%.

RHO-DAC/ATOC CONTROL OF THE G11B GYRO POSITION

ABSTRACT

The dual-mode control of the position of the electrostatic gyro G11b is considered. In the linear mode, the control law is of the type of C.D. Johnson's DAC (Disturbance Accommodation Control), in which the stability of the ideal state-space controlled plant and of the Kalman-Bucy filter (which estimates both the state of the plant and of the external disturbance) is ensured by rho-synthesis, i.e. each of these subsystems is designed to have rhobust stability by means of the Bass ρ-LQG approach. In the ATOC (Adaptive Time-Optimal Control) mode, use is made of a special case of a control law published by Bass in 1956 and independently rediscovered in 1988 by J. Haberstock & C.D. Johnson.

Introduction

Consider the accompanying block diagram of the open-loop plant. The problem is defined by 4 gains g_1 and one time-constant T_c whose values are given as follows:

g_1 = 2.58 volts/volt ,

g_2 = 200 cm/sec^2/volt ,

g_3 = 3.937×10^5 μ-inches/cm ,

g_4 = 0.048 volts/μ-inch ,

T_c = 3.26×10^{-5} sec .

As can be seen in the block diagram, $x_p \in \mathbb{R}^3$ is a 3-vector, as are its rate of change $x_v \equiv \dot{x}_p$ and acceleration $x_\alpha \equiv \dot{x}_v$. However, there are 4 non-orthogonal actuators, so that $x_A \in \mathbb{R}^4$ is a 4-vector. The external disturbance $(E_d \cdot v_d) \in \mathbb{R}^3$ is a 3-vector consisting of the first, third, and fifth components of a 6-vector $v_d \in \mathbb{R}^6$. The dimensions of these quantities are given by:

$[\,x_p\,]$ = μ-inches , $x_p \in \mathbb{R}^3$,

$[\,x_v\,]$ = μ-inches/sec , $x_v \in \mathbb{R}^3$,

$[\,x_\alpha\,]$ = μ-inches/sec^2 , $x_\alpha \in \mathbb{R}^3$,

$[\,E_d \cdot v_d\,]$ = cm/sec^2 , $(E_d \cdot v_d) \in \mathbb{R}^3$, $v_d \in \mathbb{R}^6$,

$[\,x_A\,]$ = volts , $x_A \in \mathbb{R}^4$,

$[\,y\,]$ = volts , $y^3 \in \mathbb{R}^3$, $y^4 \in \mathbb{R}^4$, $y^3 = M \cdot y^4$,

$[\,u\,]$ = volts , $u^3 \in \mathbb{R}^3$, $u^4 \in \mathbb{R}^4$, $u^4 = M^\dagger \cdot u^3$.

Letting $\dot{}\ \equiv d/dt$, where as usual t denotes time, the controller design problem can be formulated in terms of the following coupled vector-differential equations.

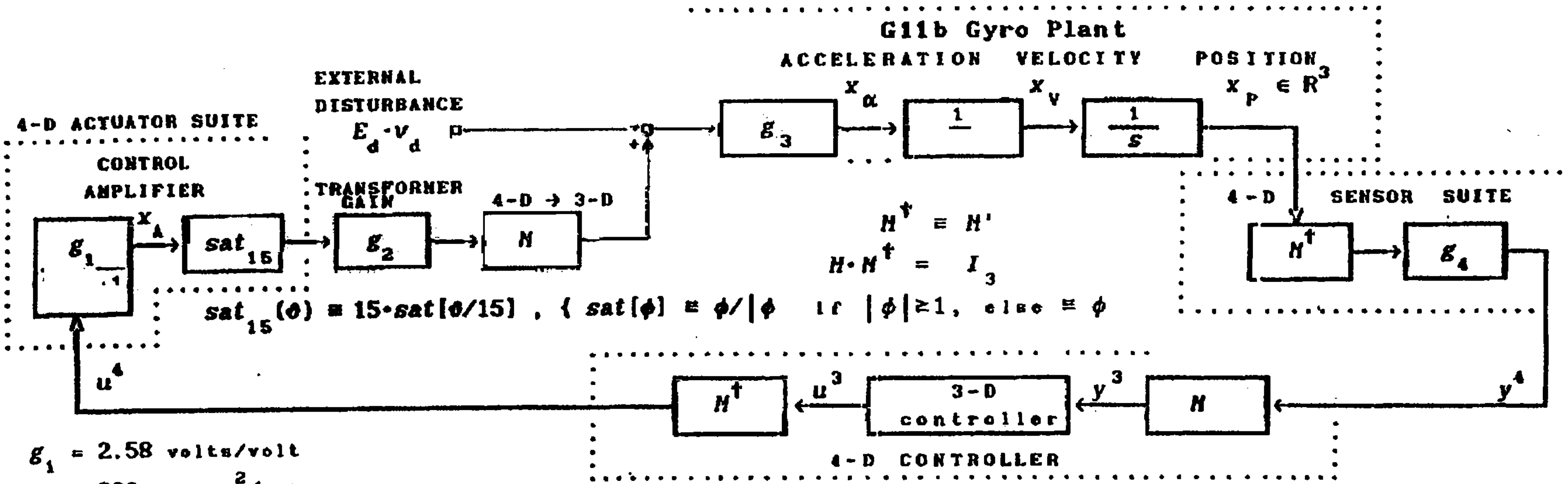
$$\dot{x}_p = x_v \,, \tag{1}$$

$$\dot{x}_v = g_2 \cdot g_3 \cdot M \cdot \{15 \cdot sat[x_A/15]\} + g_3 \cdot (E_d \cdot v_d) \,, \tag{2}$$

$$\dot{x}_A = -(1/T_c) \cdot x_A + (g_1/T_c) \cdot u^4 \,, \tag{3}$$

$$\dot{v}_d = A_d \cdot v_d \,, \tag{4}$$

$$y^4 = g_4 \cdot M^\dagger \cdot x_p \,, \tag{5}$$



g_1 = 2.58 volts/volt

g_2 = 200 cm/sec^2/volt

g_3 = 3.937x10^5 μ-inches/cm

g_4 = 0.048 volts/μ-inch

T_c = 3.26x10^{-5} sec.

s = complex frequency (Laplace variable)

G11B GYRO CONTROLLER DESIGN PROBLEM

$$M = (4/3)^{1/2}\cdot\begin{pmatrix} 0, & (3/2)^{1/2}/2, & 0, & -(3/2)^{1/2}/2 \\ 0, & -(2)^{1/2}/4, & 1/(2)^{1/2}, & -(2)^{1/2}/4 \\ 3/4, & -1/4, & -1/4, & -1/4 \end{pmatrix}$$

$$\dot{x}_p \quad x_v\,, \qquad (\in \mathbb{R}^3)$$

$$\dot{x}_v = g_2\cdot g_3\cdot M\cdot\{15\cdot sat[x_A/15]\} + g_3\cdot E_d\cdot v_d,\quad \dot{v}_d = A_d\cdot v_d \quad (\in\mathbb{R}^6)$$

$$\dot{x}_A \quad -(1/T_c)\cdot x_A + (g_1/T_c)\cdot u^4\,, \qquad (\in \mathbb{R}^4)$$

$$y^4 = g_4\cdot M^\dagger\cdot x_p\,, \qquad (\in \mathbb{R}^4)$$

x_p = μ-inches, $x_p \in \mathbb{R}^3$

[x_v = μ-inches/sec, $x_v \in \mathbb{R}^3$

. x_α = μ-inches/sec^2, $x_\alpha \in \mathbb{R}^3$

$E_d\cdot v_d$] = cm/sec^2, $(E_d\cdot v_d) \in \mathbb{R}^3$, $v_d \in \mathbb{R}^6$

x_A = volts, $x_A \in \mathbb{R}^4$

y = volts, $y^3 \in \mathbb{R}^3$, $y^4 \in \mathbb{R}^4$, $y^3 = M\cdot y^4$

u = volts, $u^3 \in \mathbb{R}^3$, $u^4 \in \mathbb{R}^4$, $u^4 = \kappa\cdot M^\dagger\cdot u^3$, ($\kappa$=1 or 15/$g_1$

where the 3x4 transformation matrix M is defined by

$$M = (4/3)^{1/2} \begin{bmatrix} 0, & (3/2)^{1/2}/2, & 0, & -(3/2)^{1/2}/2 \\ 0, & -(2)^{1/2}/4, & 1/(2)^{1/2}, & -(2)^{1/2}/4 \\ 3/4, & -1/4, & -1/4, & -1/4 \end{bmatrix}$$

If one defines the pseudo-inverse $M^{\dagger}$ of M by

$$M^{\dagger} = M' , \tag{7}$$

where ' denotes matrix transposition, then it is easy to verify by inspection that

$$M \cdot M^{\dagger} \equiv I_3 . \tag{8}$$

Model Order Reduction

The characteristic times of interest in the mechanical motion of the rotor are of the order of milliseconds. However, the actuator states have response times of the order of a few dozen microseconds. Therefore as a first approximation, we can regard the effect of the actuator dynamics as negligible, and eliminate (3) by using

$$x_A \equiv g_1 \cdot u^4 \tag{9}$$

Then the remaining equations become

$$\dot{x}_p = x_v ,$$

$$\dot{x}_v = g_2 \cdot g_3 \cdot M \cdot \{15 \cdot sat[(g_1/15) \cdot u^4]\} + g_3 \cdot (E_d \cdot v_d) ,$$

$$\dot{v}_d = A_d \cdot v_d ,$$

$$y^4 = g_4 \cdot M^{\dagger} \cdot x_p$$

Linear Case

In the case wherein none of the actuators are saturated, we may take

$$u^4 \equiv M^{\dagger} \cdot u^3 , \quad y^3 = M \cdot y^4 , \tag{14}$$

to obtain

$$\dot{x}_p = x_v , \tag{15}$$

$$\dot{x}_v = (g_1 \cdot g_2 \cdot g_3) \cdot \{ u^3 + (1/[g_1 \cdot g_2]) \cdot (E_d \cdot v_d) \} , \tag{16}$$

$$\dot{v}_d = A_d \cdot v_d , \tag{17}$$

$$y^3 = g_4 \cdot x_p \tag{18}$$

Upon inspection, this system consists of three uncoupled systems, for the (x,y,z) components of x_p . For convenience, we shall consider any one of these systems, and then triplicate the result. Define

$$\omega_d = 419.9728328 \text{ rad/sec} , \tag{19}$$

$$\alpha_0 = (\omega_d)^2 = 1.763771803 \times 10^5 \text{ (rad/sec)}^2 \tag{20}$$

$$\alpha_1 = 2 \cdot \zeta_d \cdot \omega_d = 60.0 \text{ rad/sec} , \tag{21}$$

which correspond to the measured dynamic characteristics of a shock profile. Next, define

$$B_2 = (g_1 \cdot g_2 \cdot g_3) = 2.03149606092 \times 10^8 \quad [\mu\text{-inches/sec}^2,$$

$$E_{d1} = (1/[g_1 \cdot g_2]) = 0.00193798449612 \quad [\text{volts/(cm/sec}^2)] \ , \tag{23}$$

$$C_1 = 0.048 \quad [\text{volts}/\mu\text{-inch}] \ , \tag{24}$$

and the control problem for each separate (de-coupled) axis becomes of the form

$$\dot{x} = Ax + Bu + B \cdot E_d \cdot v_d + E \cdot v(t) \ ,$$

$$\dot{v}_d = A_d \cdot v_d + B_d \cdot v(t) \ ,$$

$$y = Cx + C_d \cdot v_d + D \cdot v(t) + w(t) \ , \tag{27}$$

where $v(t)$ and $w(t)$ are zero-mean Gaussian stochastic white-noise processes of intensities σ_v^2 and σ_w^2 , respectively, and where now

$$A = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \quad C = (\ C_1\ ,\ 0\ )\ , \quad D = 0\ , \quad E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A_d = \begin{pmatrix} 0, & 1 \\ -\alpha_0, & -\alpha_1 \end{pmatrix} \quad B_d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad C_d = (\ 0\ ,\ 0\ )\ , \quad E_d = (\ E_{d1}\ ,\ 0\ )$$

The *DAC* solution to the preceding problem has the "certainty equivalence" form

$$u = -K\hat{x} - E_d \cdot \hat{v}_d \ , \tag{30}$$

where $u = -Kx$ is the ideal state feedback, assuming x known, and $u_d = -E_d \cdot v_d$ is the ideal disturbance feedforward, assuming v_d known. Here $\hat{x}$ and $\hat{v}_d$ are asymptotic estimates obtained from the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + B \cdot E_d \cdot \hat{v}_d + L \cdot (\ y - \hat{y}\ ) \ ,$$

$$\dot{\hat{v}}_d = A_d \cdot \hat{v}_d + L_d \cdot (\ y - \hat{y}\ ), \tag{32}$$

$$\hat{y} = C\hat{x} + C_d \cdot \hat{v}_d \ , \tag{33}$$

where (K, L, L_d) are chosen as follows. Define

$$A_{cl} = A - B \cdot K \ , \quad A_{mf} = A_m - L_m \cdot C_m \ , \tag{34}$$

where

$$A_m = \begin{pmatrix} A & , & B \cdot E_d \\ 0_{2X2} & , & A_d \end{pmatrix}, \quad L_m = \begin{pmatrix} L \\ L_d \end{pmatrix}, \quad C_m = (\ C,\ C_d), \quad E_m = \begin{pmatrix} E - L \cdot D, -L \\ B_d - L_d D, -L_d \end{pmatrix} . \tag{35}$$

Now use the ρ-LQG synthesis procedure explained elsewhere to choose gain matrices (K, L, L_d) such that A_{cl} and A_{mf} are Hurwitz matrices. Next, define

$$\tilde{x} = x - \hat{x} \ , \quad \tilde{v}_d = v_d - \hat{v}_d \ . \tag{36}$$

$$x_t = \begin{pmatrix} x \\ \tilde{x} \\ \tilde{v}_d \end{pmatrix} \quad v_t(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \tag{37}$$

and it is a straightforward matter of algebraic manipulation to prove [e.g. subtract (31)-(32)-(33) from (25)-(26)-(27)] that

$$\dot{x}_t = A_t x_t + B_t \cdot v_t(t) , \tag{38}$$

where

$$A_t = \begin{pmatrix} A_{cl}, & B\cdot K, & B\cdot E_d \\ 0_{4\times 2}, & & A_{mf} \end{pmatrix} , \quad B_t = \begin{pmatrix} E & , & 0_{2\times 1} \\ & & \dot{E}_m \end{pmatrix}$$

In the present case, the preceding procedure leads to

$$K = (1.16462411649233 , 0.00015210216556) , \tag{40a}$$

$$L = (1224564.0517 , 26907082848.9341) , \tag{40b}$$

$$L_d = (664531988.17 , 2393564359189.47) . \tag{40c}$$

The appended example shows that an initial 3-g shock causes $|K\hat{x}+E_d\hat{v}_d|$ to reach an amplitude of 12 volts; noting the gain $g_1 = 2.58$ volts/volt, this would saturate the 15-volt actuator unless its maximum value were raised from 15 volts to 30.96 volts. Consequently the consideration of the non-linear case becomes mandatory.

Nonlinear Case

In this case the model order reduction proceeds similarly, except that instead of (14) we must take

$$u^4 \equiv (15/g_1)\cdot M^{\dagger}\cdot u^3 , \quad y^3 = M\cdot y^4$$

which puts (11) into the form

$$\dot{x}_v = (15\cdot g_2\cdot g_3)\cdot M\cdot\{sat[M^{\dagger}\cdot u^3]\} + g_3\cdot(E_d\cdot v_d) .$$

As a heuristic approximation, we shall investigate the possibility that use of approximation

$$M\cdot sat[M^{\dagger}\cdot u^3] \approx u^3 \tag{43}$$

may prove viable when u^3 is an "einheitsvektor", i.e when each component of u^3 equals ±1 or 0 (so that a decoupled bang-bang control law for (43) is appropriate). Using (43), the system becomes three uncoupled single-channel *ATOC* problems:

$$\dot{x}_p = x_v , \quad \dot{x}_v = \ddot{x}_p = \Gamma\cdot\{ u^3 + (1/[15\cdot g_2]) (E_d\cdot v_d) \} , \quad \Gamma \equiv 15\cdot g_2\cdot g_3 , \tag{44a}$$

where it is known that the *ATOC* control law is now, for $i = 1, 2, 3,$

$$u_i^3 = sgn[\sigma_i] , \quad \sigma_i = -(x_{pi} + \kappa_i\cdot x_{vi}) , \quad \kappa_i = \kappa_i(x_v, E_d\cdot v_d) > 0 , \tag{44b}$$

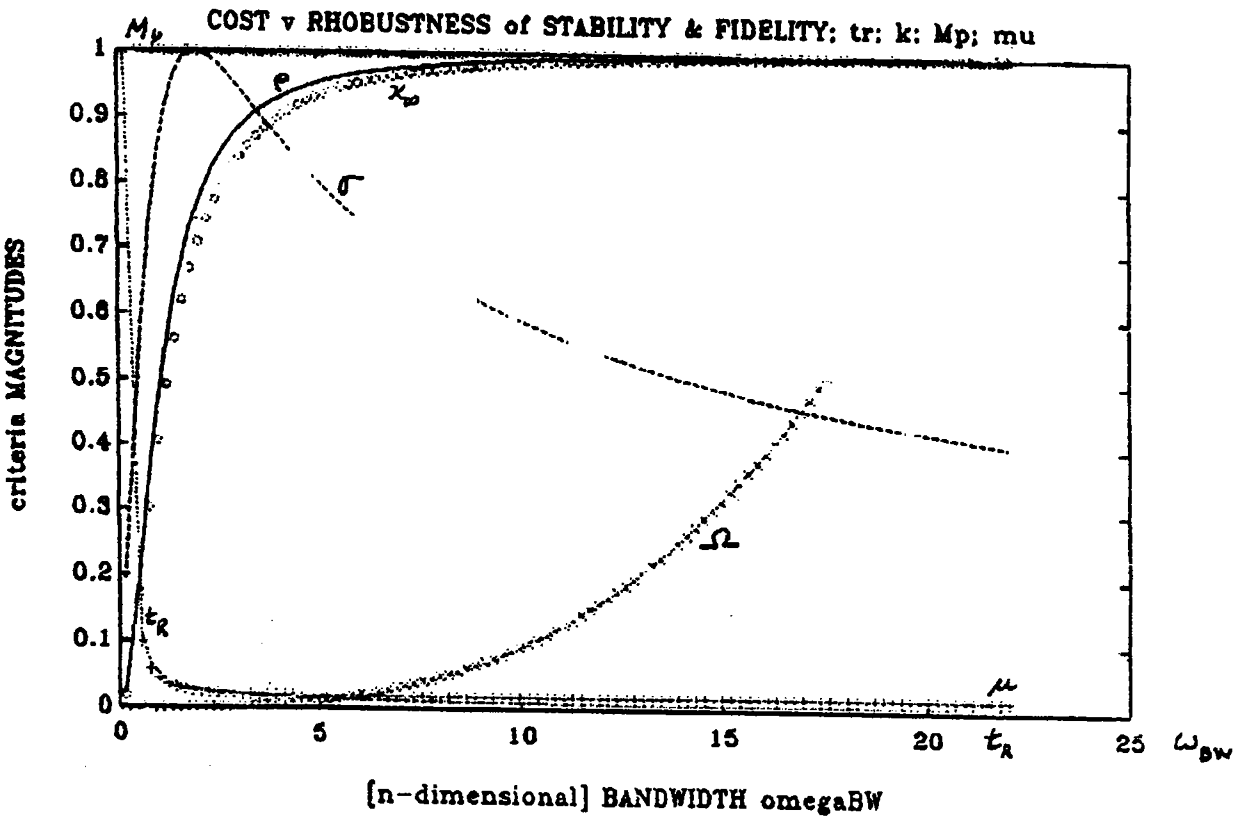
$$\kappa_i = |x_{vi}|/\{ 2\cdot\Gamma\cdot(1 - [0.99]\cdot sgn[x_{vi}]\cdot sat[(1/[15\cdot g_2]) (E_d\cdot v_d)_i]) \} . \tag{44c}$$

In principle, the *ATOC* law does not exist unless $|(1/[15\cdot g_2]) (E_d\cdot v_d)_i| < 1$; but by inserting the *sat* function we can allow brief peaks in which an oscillatory disturbance exceeds this basic requirement. The purpose of the insertion of the 0.99 factor is to prevent the "adaptive gain" κ_i from becoming infinite.

Dual Mode (Variable Structure) Control

We use the linear *ρ-DAC* control (30)-(33) + (40a,b,c) whenever $|(E_d\cdot v_d)_1| < \varepsilon$ for a criterion ε to be determined by trial-and-error; otherwise, use *ATOC* [(41) and (44)].

Rhobustness Characteristics

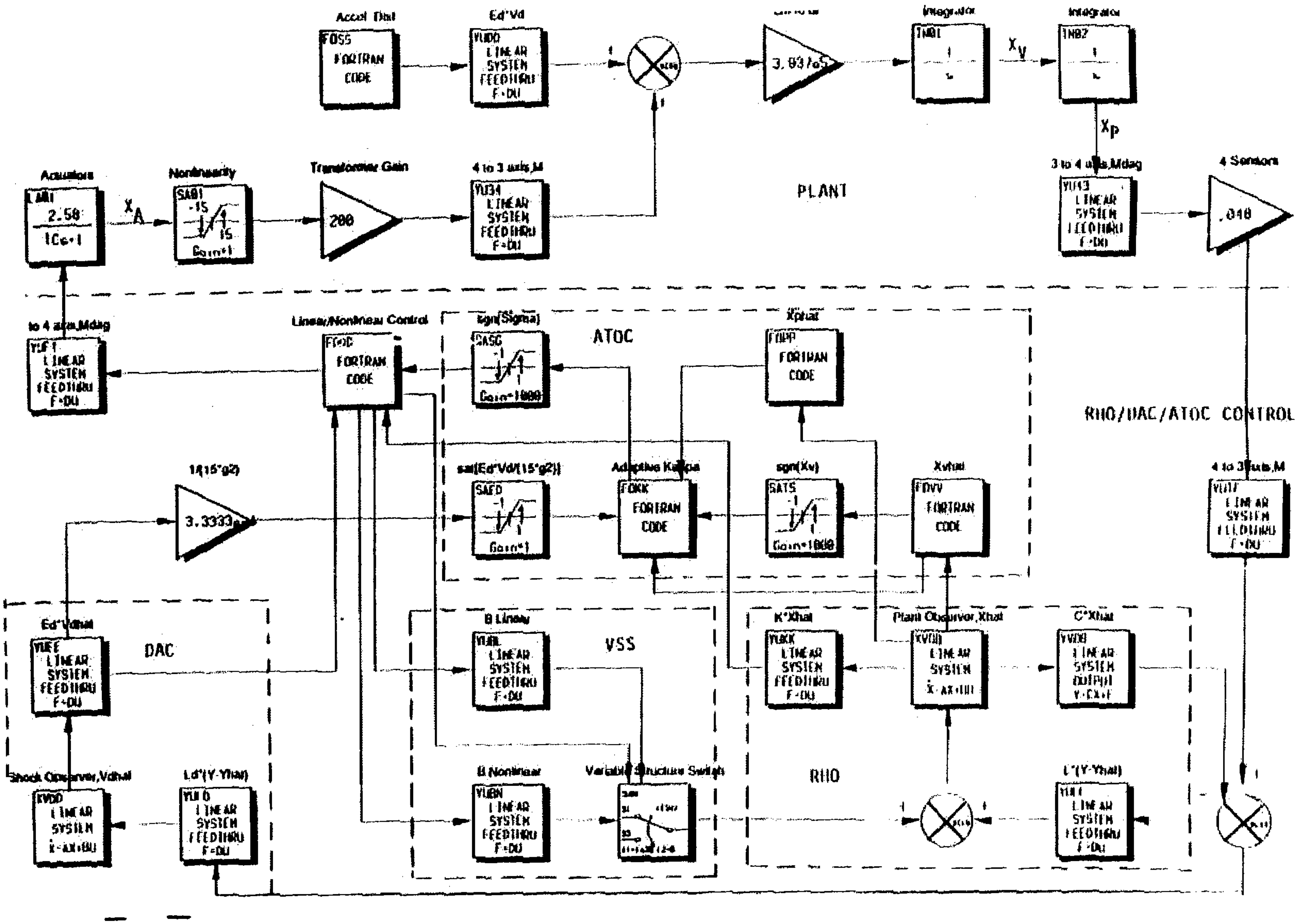


t_R	=	response time ....
ρ	=	*stability* rhobustness margin ——— (disturbance attenuation & parametric insensitivity)
σ	=	*fidelity* robustness (stochastic - - - - dispersion rejection factor)
κ_∞	=	(static) accuracy ∞∞∞
		(multivariable) Bandwidth
M_p	=	(normalized) peak resonance ⊙⊙⊙⊙⊙
μ	=	(absolute) peak resonance +++++++ M_p/κ_∞
Ω	=	(rms power) cost of control xxxxxxx

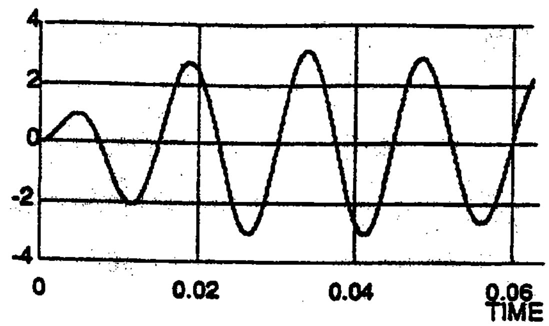
Bass Characteristics

Rho = 0.6815776	OmegaBW = 15381.57763
Sig = 0.0169286	Kappinf = 0.99999999147
Trsp = 0.000974	Omegstr = 0
Mu = 1.0000000852	Lambd = 7692.3076920
Mpeakres = 1	Gamm = 11286.0342857

Simulation Diagram Rho/Dac/Atoc Law

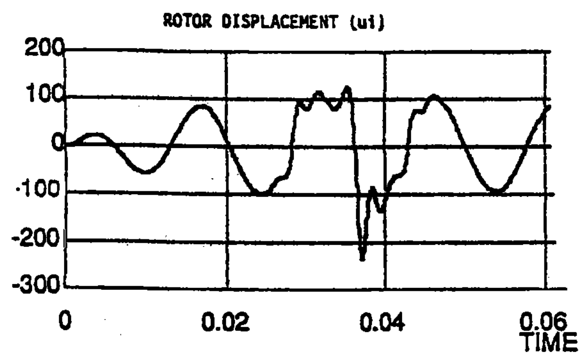


Gyro Shock Results   Single Channel

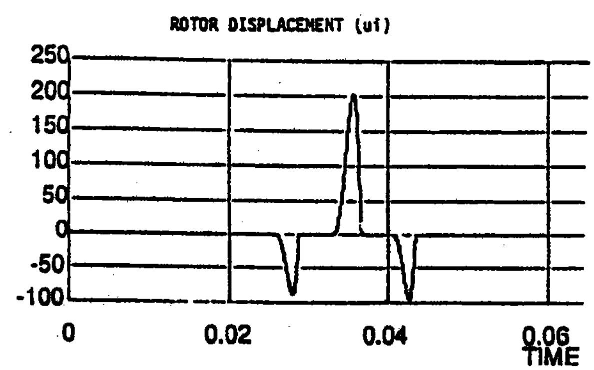


SHOCK INPUT

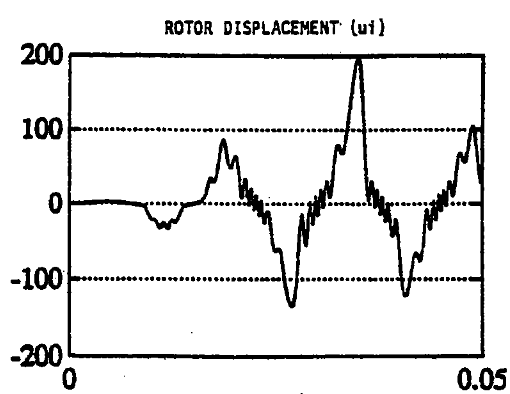
V_d = $250*t*e^{-30t}Sin(2*pi*66.7t)$ g's



EXISTING CONTROLLER



RHO/DAC/ATOC CONTROLLER



FUZZY CONTROLLER

NOTE: ROTOR DISPLACEMENT LIMIT IS±300 ui (I.E. ROTOR HITS CAVITY WALL)