DISKCOPTER SYNTHESIS THEOREM:

ELUCIDATING FIGURE-OF-MERIT OPTIMIZATION

& RESOLUTION OF LONG-STANDING PARADOX THEREON

by

Robert W. Bass,[*] Leonard W. Grayson[†]

[*]Innoventek Inc.
45960 Indian Way #612, Lexington Park, MD 20653-5338

ABSTRACT

A 52-year old technical error in the professional literature of Helicopter and VTOL Rotorcraft optimization theory is exposed and rectified. The *Figure of Merit* (*FM*) is a measure of the rotor's *lifting efficiency* defined by

$$FM = (_{\text{[induced]}}\,\textbf{Lifting Power})/(\textbf{Total Power}).$$

Since 1955 ([4],[1]) it has been stated that the "optimum" value for a helicopter or VTOL rotorcraft is $FM = 2/3$, on the basis of a mathematical "Proof." Notwithstanding that alleged "theorem" (which Stepniewski did not repeat in his 1979 book [5]), practical experts on VTOL, such as McCormick ([1], 1967) and Prouty ([3], 1986), have long known that *FM*'s in the range

$$0.75 \leq FM \leq 0.80$$

are attainable, and that *FM* is indeed a good *comparative* measure of lifting efficiency. Indeed, all later authors ([3],[5],[6],[7]) have presented a correct theory of *FM* which adequately predicts reality, and have simply ignored the cited "theorem."

This paper resolves that long-standing paradox by a careful analysis of the conceptual design of a general class of both helicopters and other VTOL rotorcraft, including Grayson's patented [8] Diskcopter [*cf.* Fig.1 and the Appendix below], culminating in a Synthesis Theorem which proves that ALL salient rotorcraft characteristics (including *Rotor Radus R*, as well as *FM*), are, in principle, uniquely determined, once the rotorcraft's *Gross Weight* W_G, the rotor's *Tip Speed* V_T, the rotor's mean blade-element profile L/D characteristics (C_ℓ, C_d), and the rotor's geometric *Solidity* σ are specified, along with, in the Diskcopter (DSK) case, the *Aspect Ratio* ξ, $0 < \xi < 1$, defined as the ratio of inner to outer radius of the cylindrical annular housing of two *non-axially-pivoted*, stacked, counter-rotating ring-fans, which have zero net angular momentum after being rotated by Grayson's patented [9] gear mechanism. The helicopter case is recovered by setting $\xi = 0$. Because Grayson has flown a man-carrying DSK with a solidity $\sigma = 0.95$, whereas helicopters are usually limited to $0.01 \leq \sigma \leq 0.05$ this means that helicopters cannot have $FM > 82\%$, whereas the DSK can have an *FM* of 97%, and for high-speed cruise-mode, *with* VTOL, is mission-optimal!

NOMENCLATURE

W_{G} = gross weight of rotorcraft

V_{T} = rotor tip-speed

C_ℓ = mean blade-element lift-coefficient

C_d = mean blade-element drag-coefficient

ρ = ambient air density

R = rotor radius

σ = rotor solidity (ratio of projected planform blade-area to $\pi\cdot(1-\xi^2)\cdot R^2$) $\geq \sigma_{\mathrm{min}} := 27\cdot(C_d)^2/(C_\ell)^3$

P_{i} = $(W_{\mathrm{G}})^{3/2}/[\{2\pi\cdot\rho\cdot(1-\xi^2)\}^{1/2}\cdot R] = [1/(2\cdot 3^{1/2})]\cdot(\sigma C_\ell)^{1/2})\cdot(1+\xi+\xi^2)^{1/2}\cdot W_{\mathrm{G}}\cdot V_{\mathrm{T}} =$

= induced power (required for lift) [derived below]

P_{p} = $(1/8)(\sigma C_d)\cdot[\rho\cdot(V_{\mathrm{T}})^3]\cdot(\pi\cdot(1-\xi^4)\cdot(1+\xi)\cdot R^2) = (3/4)\cdot(C_d/C_\ell)\cdot\{1+[\xi^3/(1+\xi+\xi^2)]\}\cdot W_{\mathrm{G}}\cdot V_{\mathrm{T}} =$

= profile drag power of rotor blade

P = total applied hovering power $\approx P_{\mathrm{i}} + P_{\mathrm{p}}$

ε = $P_{\mathrm{p}} / P_{\mathrm{i}} = \{(3\cdot 3^{1/2}/2)/[\sigma^{1/2}\{(C_\ell)^{3/2}/C_d\}]\}\cdot\{(1+\xi)\cdot(1+\xi^2)/(1+\xi+\xi^2)^{3/2}\}$ = (drag power)/(lifting power)

FM = P_{i}/P = $1/(1+\varepsilon)$ = Figure of Merit

$D.L.$ = $W_{\mathrm{G}}/[\sigma\{\pi\cdot(1-\xi^2)\cdot R^2\}]$ = Disk Loading

$T.L.$ = $\sigma\cdot(D.L.)$ = $W_{\mathrm{G}}/[\pi\cdot(1-\xi^2)\cdot R^2]$ = Thrust Loading

$L.L.$ = W_{G}/P = Lift Loading

V_{DW} = $(FM\cdot P)/W_{\mathrm{G}}$ = rotor downwash velocity

ξ = Aspect Ratio

η = Lifting Efficiency = $W_{\mathrm{G}}\cdot V_{\mathrm{T}}/P(\xi) \leq W_{\mathrm{G}}\cdot V_{\mathrm{T}}/P(0) \equiv 1/\{[1/(2\cdot 3^{1/2})]\cdot(\sigma C_\ell)^{1/2} + (3/4)\cdot(C_d/C_\ell)\} \leq$

$\leq (4/9)\cdot(C_\ell/C_d)$ if $\sigma \geq \sigma_{\mathrm{min}}$

INTRODUCTION

Let W_{G} denote the gross weight of a rotorcraft whose rotor has radius R, solidity σ, tip-speed V_{T}, and mean blade-element drag coefficient C_d, operating in air of ambient density ρ. Then it is well known ([1]-[3],[5]-[7]), (in the case $\xi = 0$) and re-derived below, that the total power P required to hover, is given approximately by

$$P = P_{\mathrm{i}} + P_{\mathrm{p}} = (W_{\mathrm{G}})^{3/2}/[(2\pi\cdot\rho)^{1/2}\cdot(1-\xi^2)^{1/2}\cdot R] + (1/8)(\sigma C_d)\,[\pi(1+\xi)\cdot(1-\xi^4)\cdot R^2]\,[\rho\cdot(V_{\mathrm{T}})^3], \qquad (1)$$

where the first term P_{i} denotes the *induced power* required for lift, while the second term P_{p} denotes *profile power* (wasted in overcoming rotor drag). [Equation (1) in the general case of $\xi > 0$ will be derived later below.]

Accordingly an obviously **physically significant** measure of the ***rotor's*** *lift efficiency* is given by the *Figure of Merit FM,* defined as the ratio of induced power to total power, namely

$$FM = P_{\mathrm{i}}/P = 1/(1+\varepsilon), \qquad \varepsilon = P_{\mathrm{p}}/P_{\mathrm{i}}\,. \qquad (2)$$

It is well known [and proved below] that ε, hence *FM,* is a constant *independent* of ρ and of both V_{T} and R, and dependent only upon (σ,ξ,C_ℓ,C_d), where C_ℓ is the rotor blade-element's mean lift coefficient; and C_d is its mean drag coefficient; in fact, it can be proved (as in *e.g.* [3],[5]-[7]) [and re-derived below in the case $\xi > 0$] that

$$\varepsilon = \{(3\cdot 3^{1/2}/2)/[\sigma^{1/2}\{(C_\ell)^{3/2}/C_d]\}\cdot\{(1+\xi)(1+\xi^2)/(1+\xi+\xi^2)^{3/2}\}, \qquad (3)$$

so that ε, and *FM*, *depend only* upon the rotor's *geometry* and *aerodynamic* characteristics.

It was stated in the early literature ([4],[1]) that, in the case $\xi = 0$, the required power P can be *minimized* by appropriate choice of radius R, and that [1] "for the optimum radius, $P_{\mathrm{i}} = 2\cdot P_{\mathrm{p}}$ [i.e. allegedly **optimum** $\varepsilon = ½$], and *FM* for the **optimum** physical rotor ... is $FM = 2/3$."

Despite an apparently convincing mathematical "proof" of this claimed (and, if true, obviously basic) optimization result, its irrelevance to engineering reality has been recognized since at least 1967 [1]. According to McCormick [11, the "foregoing analysis is possibly not too realistic ··· .In several respects *FM* is useless. As an academic exercise, it is interesting, but in practical applications we must consider the power ··· ." McCormick also states that "a typical $FM = 0.76$."

PARADOX

Although McCormick's judgment against the realism of "optimum $FM = 0.67$" is correct, the actually realistic and important physical significance of *FM* in comparison of competing designs is self-evident; hence *FM* must be retained, but the alleged proof of its optimality at $FM = 2/3$ must somehow be corrected. Prouty [3], followed by all subsequent authors ([5]-[7]) simply ignores that issue! Yet he deals with *FM* as of central importance, and demonstrates that by 1986 excellent rotors had attained $FM = 0.8$ or slightly more, and that (3) has adequate predictive value in preliminary engineering design.

RESOLUTION OF PARADOX

So what is wrong with the accepted "proof' that "optimum $FM = 2/3$"?

T'his "proof" consists in writing

$$P = (C_1/R) + C_2\cdot R^2, \qquad (4)$$

where, by inspection of (1),

$$C_1 = (W_G)^{3/2}/[(2\pi\cdot\rho)^{1/2}], \quad C_2 = (\pi/8)(\sigma C_d)[\rho\cdot(V_T)^3]. \tag{5}$$

Now, differentiating (4),

$$dP/dR = -(C_1/R^2) + 2C_2\cdot R = 0 \Leftrightarrow R = R_{OPT} \equiv [C_1/(2C_2)]^{1/3}, \tag{6}$$

whence, according to standard differential calculus, we have "proved" that as R increases from $R = 0$ to $R = +\infty$, P has a finite minimum given by

$$P_i = C_1/R_{OPT} \equiv (C_1)^{2/3}/(2C_2)^{1/3}, \quad P_p = C_2\cdot(R_{OPT})^2 \equiv P_i/2, \tag{7a}$$

$$\varepsilon_{OPT} = 1/2, \quad FM_{OPT} = 1/(1+[1/2]) = 2/3. \tag{7b}$$

The fallacy in the preceding argument lies in the assumption that C_2 is an *independent* constant. In fact we shall prove below that once (W_G,ρ) are given as fixed data, and once the rotor geometry (σ,ξ,C_ℓ,C_d) is fixed, then the *product* $V_T\cdot R$ is fixed! Consequently C_2 is not really a constant, but is inversely proportional to R^3, so that to be physically correct one must replace the term $C_2\cdot R^2$ in (4) by a term of the form C_3/R, where now C_3 actually is a constant. Then of course (4) has the form $P = C_4/R$, $C_4 = C_1+C_3$, where now C_4 actually is a constant, dependent only upon $(W_G,\rho,\sigma,\xi,C_\ell,C_d,V_T)$. This corrected relationship between P and R shows that (unless we take the weight of the blade into account), there is no R which minimizes P other than $R = +\infty$.

The fallacy in the preceding long-overlooked error in the professional literature was discovered as a result of the careful formulation and proof of the following theorem, pertaining to preliminary conceptual VTOL rotorcraft design.

ROTORCRAFT SYNTHESIS THEOREM

In order to proceed, we must recall the additional concepts of *Disk Loading D.L., Thrust Loading T.L., Lift Loading L.L.,* and rotor *downwash velocity* V_{DW} as defined in the Nomenclature above and in Prouty [3].

SYNTHESIS THEOREM. Given the basic design parameters

$$(W_G,\rho,\sigma,\xi,C_\ell,C_d,V_T) \tag{8}$$

then the remaining design and performance characteristics

$$(\varepsilon, FM, V_{DW}, D.L., L.L., P, R), \tag{9}$$

are uniquely determined and in fact given by the following formulae**:**

$$\varepsilon = \{(3\cdot 3^{1/2}/2)/[\sigma^{1/2}\{(C_\ell)^{3/2}/C_d\}]\}\cdot\{(1+\xi)(1+\xi^2)/(1+\xi+\xi^2)^{3/2}\}, \tag{10}$$

$$FM = 1/(1+\varepsilon), \tag{11}$$

$$V_{DW} = [1/(2\cdot 3^{1/2})](\sigma\cdot(1+\xi+\xi^2)\cdot C_\ell)^{1/2} V_T, \tag{12}$$

$$D.L. \;=\; T.L./\sigma, \quad T.L. \;=\; 2\rho(V_{\mathrm{DW}})^2 \;=\; (1/3)(\sigma\cdot C_\ell)q_{\mathrm{T}}, \quad q_{\mathrm{T}} \;=\; (1/2)\rho(V_{\mathrm{T}})^2, \tag{13}$$

$$L.L. \;=\; FM/V_{\mathrm{DW}}, \tag{14}$$

$$P \;=\; (1/FM)W_{\mathrm{G}}V_{\mathrm{DW}}, \tag{15}$$

$$R \;=\; (W_{\mathrm{G}}/[2\pi\cdot\rho])^{1/2}/V_{\mathrm{DW}}. \tag{16}$$

PROOF. The first item is to establish the basic formula for induced power P_{i}, which is a consequence of *conservation* of **mass**, **momentum**, and **energy** (using the standard notation that $\dot{m} = dm/dt$), namely that

$$\text{[CONSERVATION OFF MASS \& MASS-FLUX]} \qquad \dot{m} \;=\; \rho\cdot V_{\mathrm{DW}}\cdot A \;\equiv\; \rho\cdot V_{\mathrm{w}}\cdot A_{\mathrm{w}}, \tag{17a}$$

where $A = \pi\cdot(1 - \xi^2)\cdot R^2$ is the area through which ambient air of density ρ flows through the rotor with downwash velocity V_{DW}, and where $(V_{\mathrm{w}},A_{\mathrm{w}})$ are the corresponding [unknown!] final (remote) wake velocity and wake cross-sectional area far below the rotor, and conservation of mass-flux is envisioned in a flux-tube bounded by streamlines analogous to the generators of a [curvilinearl cylindrical annulus, whose top and bottom areas are respectively (A, A_{w}). Next, Newton's First Law is recalled in the form that Force F or Thrust T (in this case, equal to gross weight W_{G}) is equal to the rate of change of momentum, or momentum flux:

$$\text{[CONSERVATION OF MOMENTUM]} \quad F = T \;=\; (\dot{m}\, V)_{\mathrm{wake}} - (\dot{m}\, V)_{\text{air above rotor}} \;\equiv$$

$$\equiv\; \dot{m}\,(V_{\mathrm{w}} - 0) \;\equiv\; \dot{m}\, V_{\mathrm{w}} \;=\; (\rho\cdot V_{\mathrm{DW}}\cdot A)\cdot V_{\mathrm{w}}\,. \tag{17b}$$

Finally, recall that power is the rate at which a force does work:

$$\text{[CONSERVATION OF ENERGY]} \qquad P \;=\; (1/2)\,\dot{m}\,(V_{\mathrm{w}})^2 \;\equiv\; (1/2)\,(\rho\cdot V_{\mathrm{DW}}\cdot A)(V_{\mathrm{w}})^2 \;=$$

$$=\; F\cdot V_{\mathrm{DW}} \;\equiv\; [(\rho\cdot V_{\mathrm{DW}}\cdot A)(V_{\mathrm{w}})]\cdot V_{\mathrm{DW}}\,. \tag{17c}$$

From the two different versions of (17c), one finds that

$$V_{\mathrm{w}} \;=\; 2\cdot V_{\mathrm{DW}} \tag{17d}$$

which, when inserted into (17b), yields

$$F \;=\; T \;=\; W_{\mathrm{G}} \;=\; [2\rho\pi\cdot(1 - \xi^2)\cdot R^2]\cdot(V_{\mathrm{DW}})^2, \tag{17e}$$

which implies that

$$V_{\mathrm{DW}} \;=\; (W_{\mathrm{G}}/[2\rho\pi])^{1/2}/[(1 - \xi^2)^{1/2}\cdot R], \tag{17f}$$

which, when inserted into the first version of (17c), yields for *induced power*

$$P_{\mathrm{i}} = 2\rho\pi\cdot(1-\xi^2)\cdot R^2(V_{\mathrm{DW}})^3 = 2\rho\pi\cdot(1-\xi^2)\cdot R^2\{(W_{\mathrm{G}}/[2\rho\pi])^{1/2}/[(1-\xi^2)^{1/2}\cdot R]\}^3 \equiv (\mathrm{W}_{\mathrm{G}})^{3/2}/[(2\rho\pi)^{1/2}(1-\xi^2)^{1/2}R], \tag{18}$$

as claimed, thereby justifying the first term in (1). The result (18) is called the *momentum-flux* theory.

Further progress requires use of the *blade-element theory.* Suppose that there are N_{B} blades, each of mean chord c and length $(1-\xi)\cdot R$. Let dr denote an element of length of one blade at station r, so that the area element $dA = c\cdot dr$. If the blade tip is moving with velocity V_T, then the velocity of the element dA is $V(r) = (r/R)\cdot V_T$ because all stations on the blade have the same *angular* velocity $\omega = V_T/R$, and $V(r) = \omega\cdot r$. Let $c_\ell(r)$ denote the *lift coefficient* at the station r; it is well known that c_ℓ is a function of the (local) angle of attack, but we regard the collective pitch of the rotor blades as fixed and consider only the dependence upon r. It is also well known that the lift dT provided by the blade element dA is given by $dT = c_\ell(r)\cdot dA\cdot q(r)$, where at station r the *dynamic pressure* $q(r) = (1/2)\rho[V(r)]^2 \equiv (1/2)\rho\cdot r^2\cdot(V_T)^2/R^2 \equiv q_T\cdot(r/R)^2$, where by definition

$$q_T = (1/2)\rho\cdot(V_T)^2. \qquad (19)$$

Now the total lift T is given by integrating over r and then multiplying by N_B, according to which

$$T = W_G = N_B\int_{\xi R}^{R} c_\ell(r)\cdot dA\cdot q(r) \equiv N_B\cdot\int_{\xi R}^{R} c_\ell(r)\cdot q(r)\cdot c\cdot dr \equiv$$

$$\equiv N_B\cdot c\cdot q_T\cdot[\int_{\xi R}^{R} c_\ell(r)\cdot r^2\cdot dr]/R^2 \equiv (\pi R\cdot[1+\xi]\cdot\sigma)\cdot q_T\cdot C_\ell\cdot[\int_{\xi R}^{R} r^2\cdot dr]/R^2 =$$

$$= (1/3)\cdot\sigma\cdot C_\ell\cdot\{\pi\cdot(1+\xi)\cdot(1-\xi^3)\cdot R^2\}\cdot q_T\,, \qquad (20)$$

where in the penultimate step we have used the definition of mean as opposed to local lift coefficient, and have introduced the concept of *solidity* σ defined as the ratio of the rotor blades' area to the total disk area, namely

$$\sigma = A_B/A_R = N_B(c\cdot(1-\xi)\cdot R)/A_R = (N_B\cdot c)\cdot(1-\xi)\cdot R/[\pi\cdot(1-\xi^2)\cdot R^2] = N_B\cdot c/[\pi\cdot(1+\xi)\cdot R]. \qquad (21)$$

From (19)-(20) & (17f) we now have two separate relationships for *Thrust Loading T.L.,* namely, using the identity $(1+\xi)\cdot(1-\xi^3) \equiv (1-\xi^2)\cdot(1+\xi+\xi^2)$,

$$T.L. = W_G/[\pi\cdot(1-\xi^2)\cdot R^2] = (1/3)\cdot\sigma\,(1+\xi+\xi^2)\cdot C_\ell\cdot q_T = 2\cdot\rho\cdot(V_{DW})^2, \qquad (22)$$

where the second version comes from (17e) after squaring and rearranging (17f). We can now compute the *induced power* from the blade-element theory, namely from (17c) and (20), and (22) solved for V_{DW}, *i.e.*

$$P_i = T\cdot V_{DW} = W_G\cdot V_{DW} = (1/3)\cdot\sigma\cdot(1+\xi+\xi^2)\cdot C_\ell\cdot q_T\cdot[\pi\cdot(1-\xi^2)\cdot R^2]\cdot[\{(1/3)\cdot\sigma\cdot(1+\xi+\xi^2)\cdot C_\ell\cdot q_T\}/(2\cdot\rho)]^{1/2} \equiv$$

$$\equiv (1/[12\cdot 3^{1/2}])\cdot(\sigma\cdot(1+\xi+\xi^2)\cdot C_\ell)^{3/2}\cdot[\pi\cdot(1-\xi^2)\cdot R^2]\cdot p_T, \qquad p_T := \rho\cdot(V_T)^3. \qquad (23)$$

We next need an expression for the profile drag power, which is derived from the blade-element theory in a manner similar to that for the induced power. The drag on a blade element at station r is given by $dD = c_d(r)\cdot dA\cdot q(r)$, so the element of power loss to this drag is $dP_p = dD\cdot V(r)$, where, as before, $V(r) = \omega\cdot r = (r/R)\cdot V_T$. Accordingly, by (21),

$$P_{\mathrm{p}} = N_{\mathrm{B}}\cdot\int_{\xi R}^{R} c_{\mathrm{d}}(r)\cdot dA\cdot q(r)\cdot V(r) \equiv (N_{\mathrm{B}}\cdot c/2)\cdot\rho\cdot(V_{\mathrm{T}})^{3}\cdot[\int_{\xi R}^{R} c_{\mathrm{d}}(r)\cdot(r^{3})\cdot dr]/R^{3} \equiv$$

$$\equiv (1/2)\cdot[\pi(1+\xi)\cdot R\sigma]\cdot p_{\mathrm{T}}\cdot C_{d}\cdot[\int_{\xi R}^{R} r^{3}\cdot dr]/R^{3} = (1/8)\cdot\sigma\cdot C_{d}\cdot[\pi(1+\xi)\cdot(1-\xi^{4})\cdot R^{2}]\cdot p_{\mathrm{T}}\,. \qquad (24)$$

We have now proved the expression used in the second term of (1), thus completing the proof of equation (1).

Now use the identity $(1-\xi^4) \equiv (1-\xi^2)\cdot(1+\xi^2)$ and divide (24) by (23) to prove (3). Since (2) is just a definition, we have now verified (1)-(3) as promised, and are ready to prove the remainder of the claimed theorem. Note that (10)-(11) are just a rearrangement of (2)-(3), so we turn to (12), which, by (19) is an easy consequence of rewriting (22) in the form

$$2\cdot\rho\cdot(V_{\mathrm{DW}})^{2} = (1/6)\cdot C_{\ell}\cdot(1+\xi+\xi^{2})\cdot\rho\cdot(V_{\mathrm{T}})^{2}. \qquad (25)$$

Next, (13) was already proved in (22). Also, from the 2nd relationship in (23), $FM\cdot P \equiv P_{\mathrm{i}} = W_{\mathrm{G}}\cdot V_{\mathrm{DW}}$, which proves (15). Then using (15) and the definition of lift loading as W_{G}/P, (14) follows immediately. Finally, (16) is a corollary of (17f).

This completes the proof of (10)-(16) and therefore proves the theorem.□

Conclusions

The *FM* analysis is critical in preliminary ***comparison*** of different possible helicopter or DSK designs which are ***identical*** except for the geometric & aerodynamic parameter set $(\sigma,\xi,C_{\ell},C_{\mathrm{d}})$. Nevertheless I wish to thank Dr. J.J. McCue of the U.S. Navy Test Pilot School at Patuxent River Naval Air Station, MD, for pointing out that consideration of *FM* alone can be misleading if one attempts to compare *between* the two *categories* rather than comparing designs *within* either chosen category.

By the same kind of analysis as that detailed above, it is possible to further manipulate the preceding equations to re-state the principal results from a different perspective, namely that of pure Lifting Efficiency in terms of Gross Weight W_{G} versus required horsepower P. In particular, *Induced Power* P_{i} can be displayed as

$$P_{\mathrm{i}} = [1/(2\cdot 3^{1/2})]\cdot(\sigma C_{\ell})^{1/2})\cdot(1+\xi+\xi^{2})^{1/2}\cdot W_{\mathrm{G}}\cdot V_{\mathrm{T}}, \qquad (26)$$

while the corresponding *Profile Power* P_{p} can be displayed as

$$P_{\mathrm{p}} = (3/4)\cdot(C_{d}/C_{\ell})\cdot\{1+[\xi^{3}/(1+\xi+\xi^{2})]\}\cdot W_{\mathrm{G}}\cdot V_{\mathrm{T}}. \qquad (27)$$

The reader may readily verify the validity of the preceding formulae upon computing *FM* and verifying that the answer (2)-(3) already carefully derived during the Proof of the Rotorcraft Synthesis Theorem is recovered. Now define *Lifitng Efficiency* η as Gross Weight W_{G} divided by the required total power $P(\xi) = P_{\mathrm{i}} + P_{\mathrm{p}}$ after multiplying the numerator by tip-speed V_{T} in order to make the ratio dimensionless:

$$\eta^{-1} = P(\xi)/W_{\mathrm{G}}\cdot V_{\mathrm{T}} = [1/(2\cdot 3^{1/2})]\cdot(\sigma C_{\ell})^{1/2})\cdot(1+\xi+\xi^{2})^{1/2\cdot} + (3/4)\cdot(C_{d}/C_{\ell})\cdot\{1+[\xi^{3}/(1+\xi+\xi^{2})]\} \geq P(0)/W_{\mathrm{G}}\cdot V_{\mathrm{T}} \equiv$$

$$\equiv\ [1/(2\cdot 3^{1/2})]\cdot(\sigma C_\ell)^{1/2})\ +\ (3/4)\cdot(\,C_d/C_\ell)\ \geq\ (4/9)\cdot(C_\ell/C_d) \quad \text{if} \quad \sigma \geq \sigma_{\min}, \tag{28}$$

where the final step in (28), left to the reader, is an elementary exercise in differential calculus and curve minimization (regarding C_ℓ as the only variable), which is feasible provided $\sigma \geq \sigma_{\min} := 27\cdot(C_d)^2/(C_\ell)^3$. Since $\sigma_{\min}$ is almost always smaller than the actual σ, one could *normalize* η to be less than unity by dividing it by $(4/9)\cdot(C_\ell/C_d)$ and calling the result $\eta_n := (9/4)\cdot(C_d\,/C_\ell)\cdot\eta$, and then replace (28) by

$$\eta_n(\xi)\ \leq\ \eta_n(0)\ \leq\ 1. \tag{29}$$

The significance of (29) is that if ***VTOL Lifting Efficiency*** is the sole desideratum then ***the classical <u>helicopter</u> is mission-optimal***. But considering long-distance cruise-mode optimization, ***the* <u>DSK</u> *is mission-optimal***.

APPENDIX: the Diskcopter (DSK) VTOL Rotorcraft Configuration

For ready visualization, imagine a delta-wing geometry fixed-wing aircraft of sweep-back angle 30 degrees (corresponding to a supersonic mach-number cone of $M = 2$). Thus the vertical planform (or silhouette from above) will be that of an equilateral/isosceles triangle of side-length, say, L, and perimeter 3·L (and therefore of axial nose-to-rear length $(3^{1/2}/2)\cdot L$.

Now let this triangle be circumscribed about a circle of radius $R = L/(2\cdot 3^{1/2})$ and from elementary geometry the circle will be tangent to each side of the triangle (and in fact will touch each side of the triangle at its mid-point, namely a distance $L/2$ from the nose or one of the trailing wing-tips).

Finally, inscribe inside this circle another circle of radius ξR , $0 < \xi < 1$, and consider the relatively thin *annular disk* between the two circles which one would have if, e.g. ξ were greater than 0.75 (although 75% is NOT necessarily the optimal value of ξ, which depends upon a deeper analysis involving the relative weights of the structural and rotor materials, etc.).

Now cut (rather like Venetian blinds) openable &. closable slots, perpendicular to the direction of forward motion (in angle-of attack flight) in the fuselage both above and below the annular-disk silhouette, so that air can be sucked in from above and blown out below. Then install *non-axially pivoted* [but low friction, as with air-bearings &/or magnetic bearings!] counter-rotating cylindrically-stacked *ring-fans* to suck in air from above and blow out air below.

Notice that *inside* the central cylindrical region of diameter $2\cdot\xi\cdot R$ there is plenty of room for engines, payload, etc.

Define <u>*Figure of Merit*</u> (*FM*) as usual during VTOL (and in transition from VTOL to pure angle-of-attack flight, while the slots are gradually closed) as the power in the downdraft holding the aircraft in hovering mode divided by the actual engine power expended to drive the counter-rotating ring-fans which produce the downdraft. In a helicopter with dangerously low solidity-factor (area of rotor-sweep divided by blade area) the *FM* can be (and has been measured to be) up to 80% or thereabouts. However, it is easy to prove (both theoretically and via computer simulations) that when the geometrical & aerodynamic parameter-set (σ,ξ,C_ℓ,C_d) is chosen optimally, the *FM* can be up to 97%; furthermore, the capability of upward acceleration (per unit horsepower) provided by Grayson's novel technology, is much greater than that available from a helicopter or tilt-rotor or other design using *intrinsically inefficient axial pivoting.* The explanation lies in the fact that the inner portion of a helicopter blade contributes *relatively* low lift & high drag, whereas the <u>outer</u> portion (adjacent the near sonic-speed tip) contributes *relatively* high lift & low drag. Grayson's counter-rotating ring-fan innovation eliminates the inefficient inner portion!.

Nevertheless in fairness it should be admitted that if the mission is simply *short-haul/heavy-lift* and VTOL is <u>*required*</u>, then *the classical helicopter remains optimal*, because the power wasted in "crabbing forward" in horizontal flight, rather than using aerodynamic lift, is negligible in comparison to the other *desiderata* and the fact that pure Lifting Efficiency (gross weight per unit horsepower) <u>*is*</u> optimized at $\xi = 0$.

REFERENCES

[1] B.W. McCormick, Jr., *Aerodynamics of VSTOL Flight,* Academic Press, New York, 1967.

[2] A. Gessow & G.C Myers, Jr., *Aerodynamics of the Helicopter,* F. Ungar Pub. Co., New York, 1952 [& 1978 reprint].

[3] R.W. Prouty, *Helicopter Performance, Stability, and Control, PWS* Engineering, Boston, 1986.

[4] W.Z. Stepniewski, *Introduction to Helicopter Aerodynamics,* Chapter 4, Rotorcraft Pub. Co., Merton, PA, 1955.

[5] W.Z. Stepniewski & C.N. Keys, *Rotary Wing Aerodynamics*, NASA 1979 [Dover reprint 1984].

[6] Wayne Johnson, *Helicopter Theory*, Princeton U Press, 1980 [Dover reprint 1984].

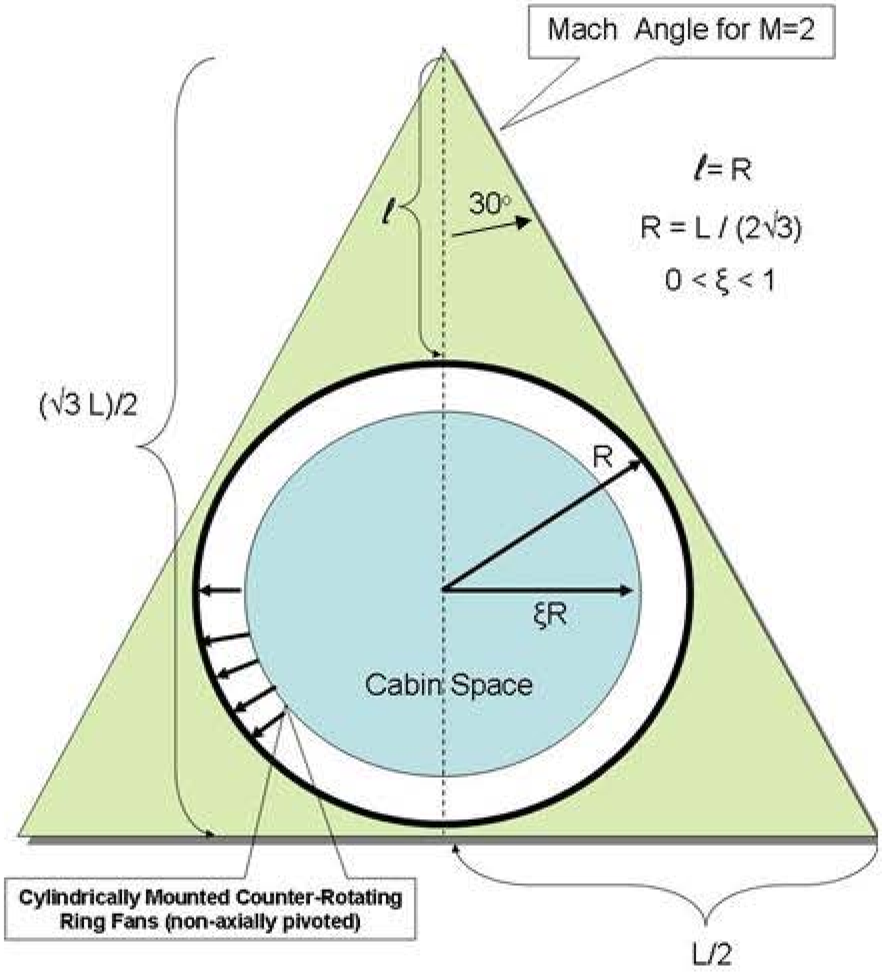
[7] J. Gordon Leishman, *Principles of Helicopter Aerodynamics*, Cambridge U Press, 2000.

[8] Leonard W. Grayson, Disc Shaped Aircraft, U.S. Patent 2,935,275 (May 3, 1960)

[9] Leonard W. Grayson, Twin Drive System, U.S. Patent 3,845,670 (November 5, 1974)

The Core Invention

Diskcopter



- More efficient than helicopter
- Faster than helicopter
- Supersonic option
- Excellent cargo volume
- Stealthy shape