



Comparison between a Stability Augmentation System (**SAS**) for **utility helicopter OH-6A** near hover when designed by **LQR optimal control** theory or by **RhoSyn maximal Stability Robustness Margin** theory.

The acronym **LQR** serves a dual mnemonic purpose:

Linear **Q**uadratic **R**egulator means an LTI system whose Performance Criterion uses minimization of a Quadratic Form in the state variables, and whose control energy used in Regulation of the system at a Set-Point is also attempted to be simultaneously minimized in a **trade-off** of *performance* versus *energy* required; moreover, the feedback gain is a unique function of two positive-definite, symmetric, weighting matrices (Q, R). If there are n state-variables, then Q is $n \times n$ and if there are m control inputs, then R is $m \times m$.



The next dozen slides briefly review the basics of RhoSyn, in terms of explaining the advantages & limitations of RhoSyn compared to MuSyn & KmSyn.

There would be no point in attempting to maximize a control system's Stability Robustness Margin unless **very significant advantages** can be **demonstrated** to accrue, preceding which we must review the basic concepts which led Bass in 1956 to introduce this new definition of a MIMO stability margin in a Brooklyn Polytechnic Symposium paper.



Consider for non-negative times $t \geq 0$ the completely general [***nonlinear, time-variant***] dynamical system, with **external forcing**

$$[d + g(t)], \quad g(0) \equiv 0, \quad \|d + g(t)\| \leq \delta,$$

bounded by some positive $\delta > 0$, namely

$$dx/dt = f(t,x) + d + g(t), \quad x(0) = x^0, \quad f(t,0) \equiv 0,$$

where x is an n -vector and where f is “smooth” in the sense of being continuously differentiable with respect to all arguments, for $0 \leq t \leq T$.

- Then in any compact (closed & bounded) neighborhood of the origin $x = 0$ there will exist a positive bound $\kappa_1 > 0$ and a constant matrix A , and a matrix $\delta A(t)$, continuous in time t , and a continuous nonlinear matrix function $\Delta(t,x)$, with $\Delta(t,0) \equiv 0$, and a *Lipschitz* constant $\kappa_2 > 0$, such that in said neighborhood, and for all t in the finite interval $[0, T]$,

$$f(t,x) \equiv \{ A + \delta A(t) + \Delta A(t,x) \} \cdot x.$$

$$\|\delta A(t)\| \leq \kappa_1, \quad \|\Delta A(t,x)\| \leq \kappa_2, \quad \|f(t,x) - A \cdot x\| / \|x\| \leq \kappa \equiv \kappa_1 + \kappa_2$$

- Accordingly, the best Linear Time-Invariant (**LTI**) approximation of this dynamical system near to its unforced equilibrium solution $x \equiv 0$ is

$$dx/dt = A \cdot x, \quad x(0) = x^0.$$



- Then there are only **SEVEN possible** kinds of **disturbances** or **perturbations** to be considered:

$$dx/dt = A \cdot x + v(t), \quad x(0) = x^0.$$

$$v(t) = v_\delta + v_i + v_e + v_s + v_h + v^0,$$

- **impulsive:** $v_\delta = x^0 \cdot \delta(t),$
- **bounded internal:** $v_i = [\delta A(t) + \Delta A(t, x)] \cdot x,$
- **bounded external:** $v_e = d + g(t), \quad \|d + g(t)\| \leq \delta,$
- **stochastic:** $E \{v_s(t)\} \equiv 0, \quad E\{v_s(t) \cdot v_s(\tau)^T\} \equiv Q \cdot \delta(t - \tau), \quad Q = Q^T \geq 0,$
- **harmonic:** $v_h = v^0 \cdot \text{Re}\{\exp(j \cdot \omega \cdot t + \theta)\},$
- **constant:** $v^0 = d \neq 0,$
- **singular perturbations:** **changing** the state-vector dimension n
[in the case of LTI systems this involve ***pole-zero cancellations***]



- Characteristics of an **asymptotically stable** (or “Hurwitz”) **LTI system**:
If all **eigenvalues** of A , or **poles** of the **resolvent** $(s \cdot I_n - A)^{-1}$ of A , have negative real parts, then there is an infinitude of non-unique pairs of numbers (λ, γ) ,
 $0 < \lambda < +\infty$, $\gamma \geq 1$, such that, for all $t \geq 0$, $\|\exp(A \cdot t)\| \leq \gamma \cdot \exp(-\lambda \cdot t)$.

- Definition** (Bass, 1956).
 $\rho = \max\{ (\lambda/\gamma) \mid \text{all } (\lambda, \gamma) \} \equiv$ stability-robustness margin, or **rhobustness**.

The (λ, γ) such that $\rho = (\lambda/\gamma)$ provide the **damping factor** λ & **overshoot factor** γ .

- Theorem** (Bass, 1991). The rhobustness factors $\lambda = \lambda(A)$, and $\gamma = \gamma(A)$ are given uniquely by

$$\lambda = 1/t_R, \quad \gamma = \exp(1) \cdot \|\exp(A \cdot t_R)\|,$$

where t_R is the unique time that maximizes $\{ t \|\exp(A \cdot t)\| \mid t \geq 0 \}$ and provides the system’s e-folding **response time**.

- Setting aside impulsive, stochastic, and singular perturbations, and considering only the four remaining possible kinds of perturbations, the most general situation to be considered is

$$\begin{aligned} dx/dt &= \{ A + \delta A(t) + \Delta A(t, x) \} \cdot x + d + g(t), \quad x(0) = x^0, \\ \|\delta A(t) + \Delta A(t, x)\| &\leq \kappa, \quad \|d + g(t)\| \leq \delta, \text{ for some positive constants } (\kappa, \delta). \end{aligned}$$



- **Theorem** (Bass, 1956). Consider a completely general dynamical system

$$dx/dt = A \cdot x + f(t,x) + d + g(t), \quad x(0) = x^0, \quad f(t,0) \equiv 0,$$

$$\|f(t,x)\|/\|x\| \leq \kappa, \quad \|d + g(t)\| \leq \delta,$$

whose unperturbed dynamical coefficient matrix A is Hurwitz, and assume that, in terms of its robustness $\rho = \rho(A)$,

$$\kappa < \rho,$$

and define the system's **marginal damping factor**

$$\lambda_* = \lambda \cdot [1 - (\kappa/\rho)],$$

and **marginal response-time**

$$t_{R*} = t_R / [1 - (\kappa/\rho)],$$

in terms of the unperturbed system's characteristics $\lambda(A)$, $t_R \equiv 1/\lambda(A)$, $\gamma(A)$, and $\rho(A) \equiv \lambda(A)/\gamma(A)$. Then, for all times $t \geq 0$ and all initial states x^0 ,

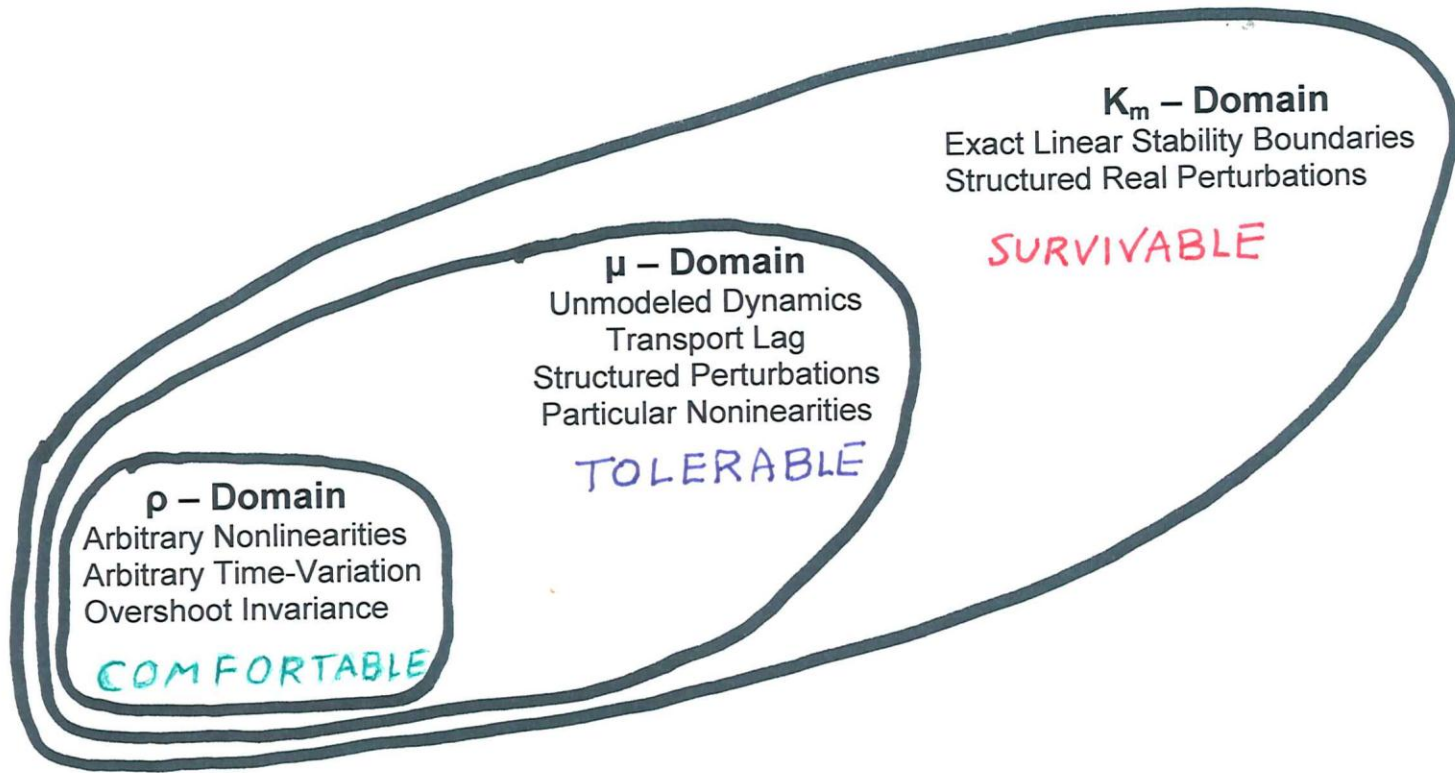
$$\|x(t)\| \leq \gamma \cdot \|x^0\| \cdot \exp(-\lambda_* \cdot t) + \{ \delta / (\rho - \kappa) \} \{ 1 - \exp(-\lambda_* \cdot t) \}.$$

Notice that the **larger** is ρ , the **smaller** is the effect of the internal & external disturbances & perturbations on both the transient & steady-state response of the perturbed system. Also it can be **proved** by counter-examples, wherein all of the preceding inequalities become equalities, that this is the **sharpest possible result** of its type, and cannot be improved. Amazingly, the overshoot factor γ remains **invariant!**

Strengths & Weaknesses of *RhoSyn* Versus *MuSyn* & *KmSyn*



- For given strictly LTI systems, K_m is the reciprocal of μ , though *MuSyn* has been generalized to include unmodeled dynamics, transport lags and some particular nonlinearities.
- However, *Rho* allows consideration of arbitrary nonlinearities, arbitrary time-variation, and arbitrary internal & external disturbances, and yet demands that the closed-loop system shall have maximum-overshoot ***invariance*** no matter what manner of uncertainty is being considered.
- But this **extreme-generality** strength of *RhoSyn* has as a consequence a weakness, which is the crudity of **norm-bounded** uncertainties versus the **finely-structured** uncertainties considered in *MuSyn*, which allows the near equivalent of considering one loop at a time and the strength of formulation of uncertainties in the **frequency-domain** rather than the time-domain, and therefore permits utilization of concepts & practices from classical control theory even in connection with high-dimensional state-vector time-domain MIMO models.
- It is possible that by suitable use of re-scaling of the variables and judicious use of weighting-factors in assigning relative importance to different variables, the domain of *RhoSyn* can be enlarged to a significant fraction of the size of the domain of *MuSyn*, but this hope presently remains a work in progress rather than an assured *fait accompli*. For SISO systems, the rescaling problem has been definitively resolved, but for general MIMO systems the usual **physical** scalings often produce examples, such as that below, wherein the *Rho* Domain is about half the size of the *Mu* Domain.



Strengths & Weaknesses of **Modern** Versus **Classical** Control Laws Designs



- The truly splendid & excellently readable 2003 book by Brian L. Stevens & Frank K. Lewis, on *Aircraft Control & Simulation*, strikes Robert W. Bass as “containing virtually everything I would be likely to think of, ever, or to learn from the recent technical literature, with just one giant exception...”
- To quote Stevens & Lewis on page 403, regarding design of Stability Augmentation Systems (**SAS**) and autopilots: “Using classical control theory, we were forced to take a one-loop-at-a-time approach to design multivariable **SAS** and autopilots. In this section we will select a performance criterion that reflects our concern with closed-loop stability and good time-responses, and then derive matrix equations that may be solved for ***all the control gains simultaneously***. ... Once the performance criterion has been selected, the control gains are explicitly computed by matrix design equations, and closed-loop stability will generally be guaranteed. This means that ***the engineering judgment in modern control enters into the selection of the performance criterion***. Different criteria will result in different closed-loop time responses and robustness properties.”
- This **one possibility for improvement** resides in the fact that they and the numerous illustrious authorities whom they cite have become accustomed to defining ***quadratic performance criteria*** which OMIT ***cross-product terms*** between the state-variables and the input variables, but in truth it is vitally important to consider a matrix triad (Q, R, S), where not only are $Q = Q^T \geq 0$ and $R = R^T > 0$ required to be symmetric non-negative definite & positive-definite matrices, respectively, but it is crucial to further allow an $S \neq 0$ and require that $dQ \equiv (Q - S \cdot R^{-1} \cdot S^T) = dQ^T > 0$ should be positive definite if one wants to optimize the closed-loop system’s robustness as measured in RhoSyn.
- Bass hastens to add that, using the slick trick of replacing A by $(A - B \cdot R^{-1} \cdot S^T)$ and replacing Q by $Q - S \cdot R^{-1} \cdot S^T$, re-enables **use** of their displayed formulae & programs!



- Other important **Performance Criteria** for systems

$$dx/dt = A \cdot x, \quad \text{with } A \text{ Hurwitz}$$

- **Fidelity Robustness Margin** [or stochastic-disturbance **dispersion-rejection** factor]

$$\sigma = \sigma(A) \equiv \{[2 \cdot \lambda(A)]^{1/2}\} / \gamma(A),$$

- **Static Loop Gain** [or reciprocal **accuracy**]

$$\kappa^\infty \equiv 1 / \|A^{-1}\|,$$

- **Bandwidth**

$$\omega_{\text{BW}} \equiv \{\|A\| / \|A^{-1}\|\}^{1/2},$$

where, defining

$$\Phi(j\omega) \equiv (j\omega I_n - A)^{-1}, \quad \Psi(j\omega) \equiv \Phi(j\omega) \cdot \Phi(j0)^{-1},$$

$$\|\Psi(j\omega)\| \geq 1/\sqrt{2}, \quad (0 \leq \omega \ll \omega_{\text{BW}}),$$

$$\|\Psi(j\omega)\| \leq 1/\sqrt{2}, \quad (\omega_{\text{BW}} \ll \omega < +\infty).$$

- **Peak Resonance Magnification Factor**

$$M_p \equiv \|A\| / \rho \geq \|\Psi(j\omega)\|.$$



RhoSyn Review:

We are given an LTI possibly MIMO system in state-vector form

$$dx/dt = A \cdot x + B \cdot u, \quad (u = -K \cdot x, \quad K = ?),$$

and desire to find a state-feedback **regulator gain** matrix K such that the resultant **closed-loop** system $dx/dt = A_{cl} \cdot x$, $A_{cl} \equiv A - B \cdot K$, is a Hurwitz matrix [i.e. the poles of its Laplace transform resolvent $(s \cdot I_n - A_{cl})^{-1}$ are all in the left half of the complex s-plane].

The **FIRST TRICK** is to remember to **pretend** that the system actually has as its dynamical coefficient matrix not A but $A_\eta = (A + \eta \cdot I_n)$, where $\eta > 0$ is an arbitrary design parameter & I_n denotes the n -dimensional identity matrix.

The choice of η is made as follows. If K is so well chosen that $A_{\eta cl} \equiv A_\eta - B \cdot K$ is a Hurwitz matrix, then obviously

$$\max\{\text{real}(\text{eig}(A_{\eta cl}))\} = \max\{\text{real}(\text{eig}(A_{cl}))\} + \eta < 0$$

which implies that $\max\{\text{real}(\text{eig}(A_{cl}))\} < -\eta < 0$ and so for all times $t \geq 0$

$$\|\exp_m(A_{cl} \cdot t)\| \leq \gamma \cdot \exp(-\eta \cdot t)$$

for some $\gamma \geq 1$.

This means that the e-folding Response Time $t_R \leq 1/\eta$.

Accordingly when we search over the design parameter η we could alternatively regard it as searching over the design-parameter t_R .



The SECOND TRICK is to use Kalman's original 1960 LQR theory without neglecting the cross-product terms (which had become habitual within a decade). So what we want to minimize is the integral over all future time of the quadratic form in (x, u) defined by [**recall: now** $dx/dt = A_\eta \cdot x + B \cdot u$!]

$$\Phi \equiv \int_0^\infty \left(\| dx/dt + \eta \cdot x \|^2 + \|\eta \cdot x\|^2 + \|B \cdot u\|^2 \right) dt$$

where the purpose of the **third term** is to prevent the control action from becoming too great, whereas the purpose of the **first term** is to attempt to force the system to behave as much like one wherein $A_{c1} = -\eta \cdot I_n$, and the purpose of the second term is to prevent the first term from becoming small (in the integral-mean-square sense) in a pathological way in which both dx/dt AND $\eta \cdot x$ can be undesirably large, yet almost opposite to each other, which would defeat the design objective.

After a large amount of straightforward but lengthy algebra, it turns out that one may write, using $\xi \equiv [x ; u]$, and letting ' denote vector-matrix row-column transposition,

$$\Phi \equiv \xi' \cdot P \cdot \xi, \quad P \equiv \begin{bmatrix} Q & S \\ S' & R \end{bmatrix},$$

where, using $A_{2\eta} \equiv A + 2 \cdot \eta \cdot I_n$,

$$Q \equiv A_{2\eta}' \cdot A_{2\eta} + \eta^2 \cdot I_n \equiv A' \cdot A + 2 \cdot \eta \cdot (A + A') + 5 \cdot \eta^2 \cdot I_n,$$

$$R \equiv 2 \cdot B' \cdot B,$$

$$S \equiv A_{2\eta}' \cdot B.$$



BUT, according to the Calculus of Variations, there is no unique minimum solution $u = -K \cdot x$ to the problem of minimizing the integral of Φ UNLESS $P = P' > 0$ is a positive-definite matrix. Therefore I consider that my greatest contribution is to have persevered with the algebra long enough to prove that the necessary & sufficient condition for this, namely that

$$\Gamma \equiv Q - S \cdot R^{-1} \cdot S' = \Gamma' > 0$$

is satisfied by virtue of the identity

$$\Gamma \equiv (1/2) \cdot \{A_{2\eta}' \cdot (I_n + E_u) \cdot A_{2\eta}\},$$

where projection matrix $E_u \equiv E_u' \equiv (E_u)^2 \geq 0$ is defined by

$$R_o \equiv B' \cdot B = R_o' > 0, \quad E_m \equiv B \cdot R_o^{-1} \cdot B', \quad E_u \equiv I_n - E_m.$$

Expanding the preceding, it is clear that Γ is the sum of various non-negative definite matrices plus a positive scalar multiple of I_n , and so positive definite. Accordingly there is a unique solution K for each $\eta > 0$.

RhoSyn



Let us consider the problem of design of an SAS for the OH-6A small utility helicopter near hover, such as is done using the conventional LQR procedure in Chapter 14 of Stanford Prof. Arthur Bryson's truly INVALUABLE book *Control of Spacecraft and Aircraft*.

F =

-0.0257	0.0113	0.0130	-.3216	0.0004	-0.0006	-0.0081	0
-0.0422	-0.3404	0.0001	-.0093	-0.0440	0.0147	0.0005	.0171
1.2600	-0.6000	-1.7645	0	-0.2600	0.0719	0.3763	0
0	0	0.9986	0	0	0.0532	0	0
0.0158	-0.0194	-0.0084	0	-0.0435	0.0034	-0.0134	.3216
-2.6200	3.1000	-0.1724	0	-0.1700	-0.8645	-1.0750	0
0.0300	-0.1900	-1.1360	0	-4.6200	-0.2873	-4.9200	0
0	0	-0.0015	0	0	0.0289	1.0000	0

G =

0.0860	0.0216	-0.0028	-0.003
-0.0016	-0.7343	0.0011	-0.003
-7.4080	-0.7850	0.3500	-0.096
0	0	0	0
0.0038	-0.0057	0.0514	0.153
0.4930	9.5070	1.9820	-25.680
1.8740	1.2060	12.7900	-0.781
0	0	0	0

K =

.791	.107	-.187	-.479	.275	.015	.004	.111
.074	-.573	-.010	-.025	.110	.327	-.007	.024
-.248	.023	.017	.146	.625	.053	.081	.446
.151	-.321	-.006	-.023	.086	-.911	.036	.036



This system is slightly unstable because its open-loop poles are
[henceforth using suffixes **LQR** or **QR** or **qr** as abbreviations, versus **rh** for **rho**]

$$\text{OpenLoopPolesLQR} = \text{eig}(F_{qr})$$

=

-4.9262

-2.0103

-0.8209

0.0143 + 0.5123i

0.0143 - 0.5123i

-0.0001 + 0.4082i

-0.0001 - 0.4082i

-0.2294



Choosing Q & R to be the identity matrices of size 8-by-8 and 4-by-4, the standard LQR procedure provides the following feedback control gain matrix K_{qr} and resultant closed-loop system

$K_{qr} =$

0.7910	0.1070	-0.1870	-0.4790	0.2750	0.0150	0.0040	0.1110
0.0740	-0.5730	-0.0100	-0.0250	0.1100	0.3270	-0.0070	0.0240
-0.2480	0.0230	0.0170	0.1460	0.6250	0.0530	0.0810	0.4460
0.1510	-0.3210	-0.0060	-0.0230	0.0860	-0.9110	0.0360	0.0360

$F_{clQR} = F - G * K_{qr} =$

-0.0956	0.0136	0.0293	-0.2795	-0.0236	-0.0115	-0.0080	-0.0087
0.0141	-0.7620	-0.0076	-0.0287	0.0368	0.2520	-0.0046	0.0345
7.2791	-0.2960	-3.1642	-3.6214	1.6531	0.3337	0.3755	0.6885
0	0	0.9986	0	0	0.0532	0	0
0.0029	0.0249	-0.0077	-0.0023	-0.0892	0.1419	-0.0231	0.2929
0.6557	0.2059	-0.1729	-0.4062	-0.3816	-27.4802	-0.2465	-0.2424
1.7483	-0.2444	-0.9956	-0.9575	-13.1946	-2.0991	-5.9269	-5.9132
0	0	-0.0015	0	0	0.0289	1.0000	0



so the LQR SAS system's closed-loop poles become:

$$\text{ClosedLoopPoles}_{\text{QR}} = \text{eig}(\text{Fcl}_{\text{qr}})$$

=

-27.4984

-4.8174

-2.0891

-0.6523 + 0.9023i

-0.6523 - 0.9023i

-0.5197 + 0.7618i

-0.5197 - 0.7618i

-0.7692



help rhosyn

use:

$[K_{tr}, K_{rh}, \eta_{tr}, w_{bss}, \eta_{atr}, w_{sstr}, \eta_{afi}, \eta_{achi}, P, P_{tr}, t_{lm}] = \text{rhosyn}(A, B, t_R);$

inputs:

A = dynamical coefficient matrix [continuous time]

B = actuator kinematics matrix

t_R = desired Response Time such that $[\|x(t_R)\|/\|x(0)\|] < 1/e$ for all $t > t_R$

outputs:

K_{tr} = gain matrix such that $A_{cl} = A - B.K_{tr}$ minimizes t_R

K_{rh} = gain matrix such that $A_{cl} = A - B.K_{rh}$ maximizes rho

η_{tr} = value of design parameter eta which maximizes rho

w_{bss} = Wiberg-Bass Characters of A_{cl} for K_{rh}

η_{atr} = value of design parameter eta which minimizes t_R

w_{sstr} = Wiberg-Bass Characters of A_{cl} for K_{tr}

η_{afi} = value of design parameter eta which maximizes sig

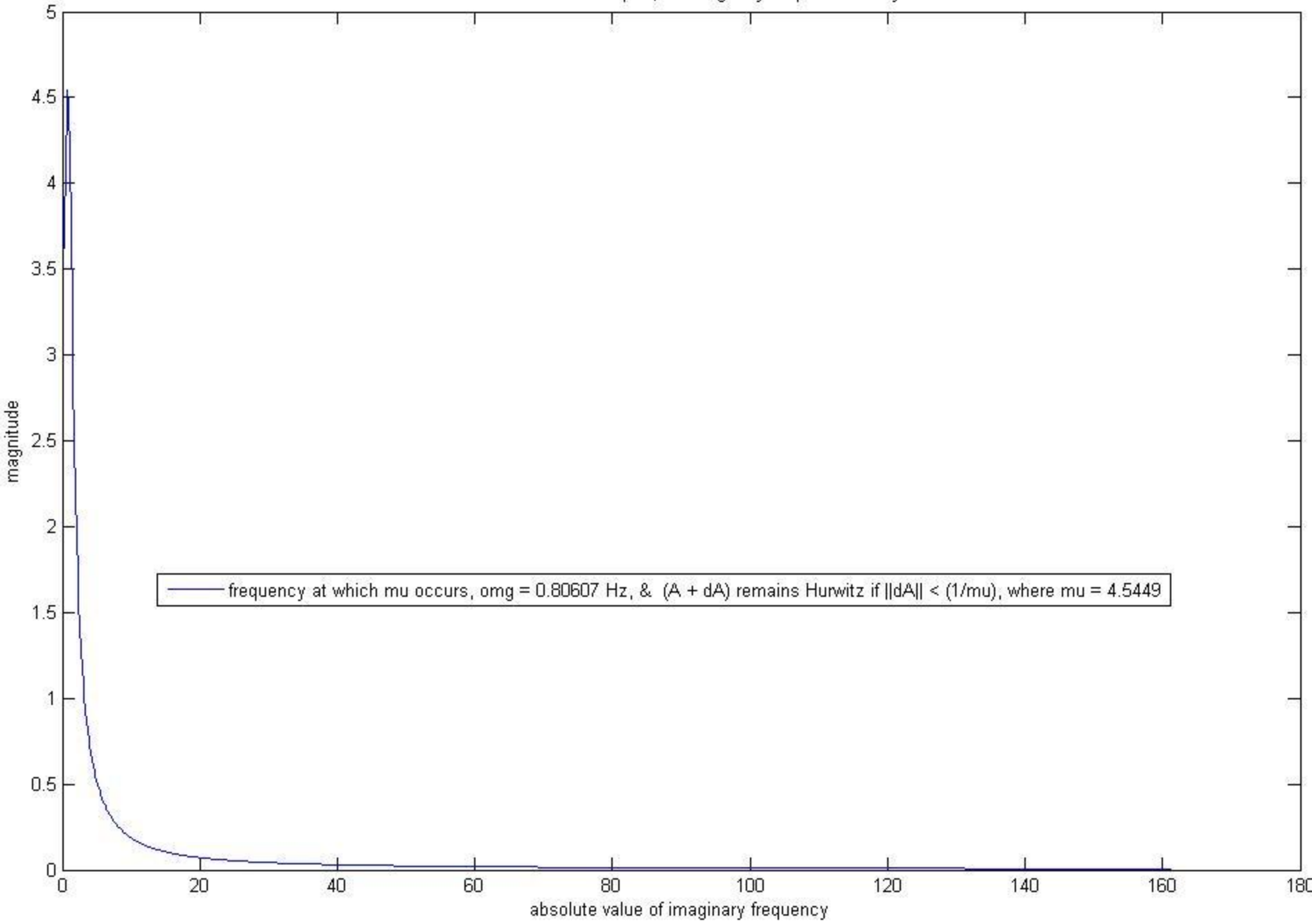
η_{achi} = value of design parameter eta which maximizes chi

P = positive-definite matrix used in finding w_{bss}

P_{tr} = positive-definite matrix used in finding w_{sstr}

t_{lm} = time-limit used in finding w_{bss}

n-dimensional Bode plot, for imaginary frequencies only





Bandwidth of a MIMO System (Bass, 1983)

Definition sharpened 3/11/07 after a valid criticism by **Friedland**, *Control System Design*, 1986, page 187.

Consider a stable LTI system (A,B,C) in continuous time, where $\|A\|$ & $\|A^{-1}\|$ denote the Euclidean norms of the dynamical coefficient matrix A and its inverse A^{-1} . Then, in terms of $\|A\|$, $\|A^{-1}\|$, $\|B\|$, $\|C\|$, $\|A^{-1} \cdot B\|$, & $\|C \cdot A^{-1} \cdot B\|$, it is easy to define frequencies $\omega_p < 1 / \|A^{-1}\|$, & $\omega_s > \|A\|$ such that if s denotes the complex Laplace variable and if one defines

$$\Phi(s) = C \cdot (s \cdot I_n - A)^{-1} \cdot B / \|C \cdot A^{-1} \cdot B\|,$$

$$\omega_{BW} = [\omega_p \cdot \omega_s]^{1/2},$$

then $0 < \omega_p < \omega_{BW} < \omega_s < +\infty$, and

$$\|\Phi(s)\| > 1/2^{1/2} \quad \text{if} \quad 0 < |s| < \omega_p,$$

$$\|\Phi(s)\| < 1/2^{1/2} \quad \text{if} \quad 0 < \omega_s < |s|.$$



In using RhoSyn, notice that as the design parameter η is varied, the closed-loop system's MIMO Bandwidth, **omegaBW**, as defined by Bass in 1983, and later refined after a valid criticism by Friedland, turns out to be a **monotonic** function of η ! Therefore everything can be plotted against omegaBW instead of η , which supplies more physically meaningful information.

So the designer can feel as if he is adjusting the system's MIMO Bandwidth to **maximize** the system's Stability Robustness Margin, ρ .

And ρ is indeed a MARGIN because

$$\|dF\| < \rho(F) \rightarrow \rho(F + dF) \geq \rho(F) - \|dF\| > 0.$$



Moreover, it turns out that the Cost of Control, or integrated Control Power, herein called Omega, is a **cubic** function of this proposed MIMO Bandwidth **omegaBW**, .

Since Newton, Gould & Kaiser thought that such a proposition was one of the most significant results of classical SISO system theory, this lends support to the plausibility of the new MIMO Bandwidth concept as defined herein!



Notice in the next two slides that for $0 < \eta < 0.75$, in which either **rho** or **sig** could be optimized, there is NO resonance whatsoever! That is, $M_p \equiv 1$.

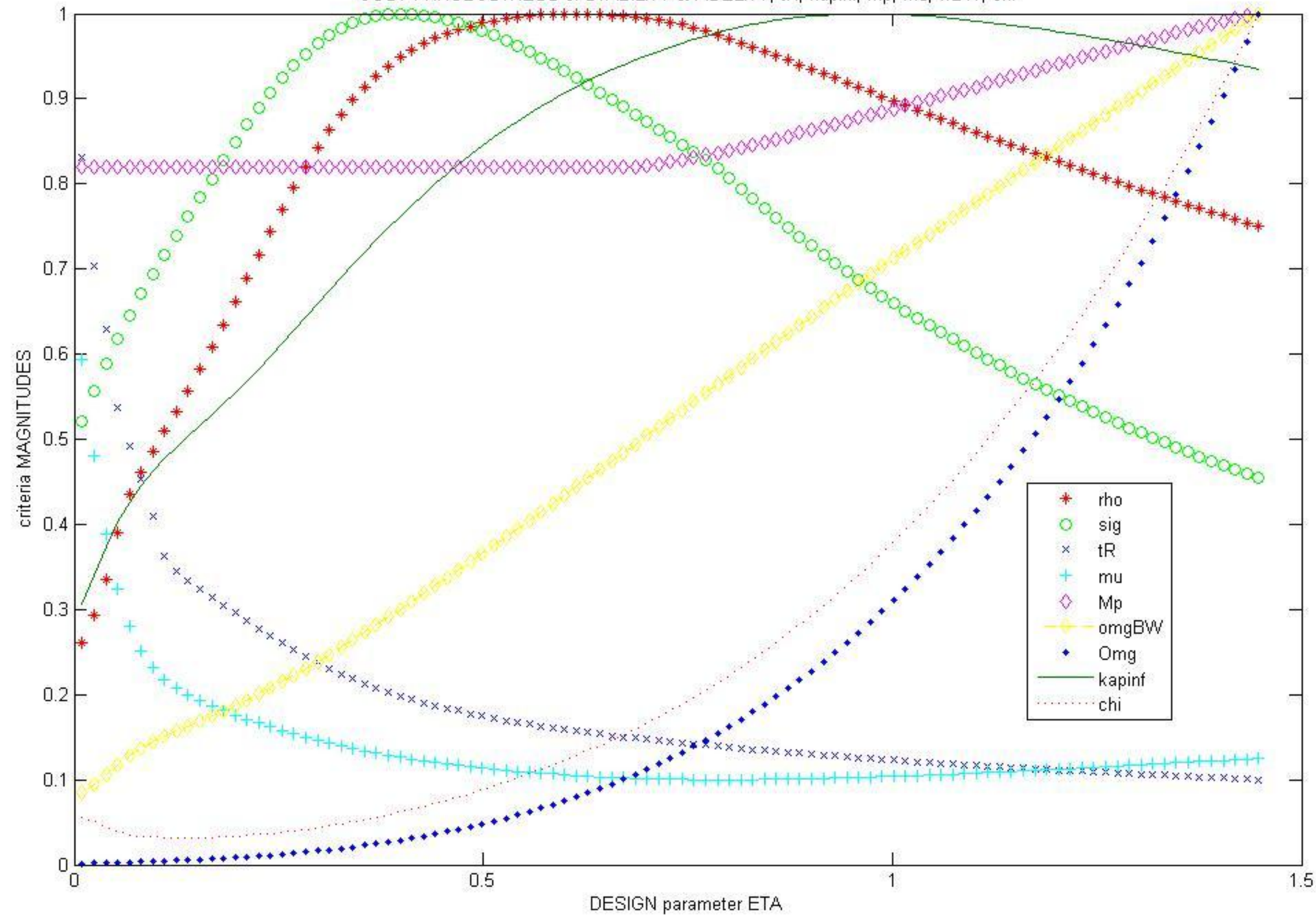
Accordingly, in this SAS system, if designed by RhoSyn, there is **zero** possibility of excitation of a resonant perturbation!

Remember that in 3 separate accidents, the rudder of the Airbus 300 has inexplicably oscillated with exponentially increasing amplitude until the rudder was torn off in flight (resulting in hundreds of fatalities).

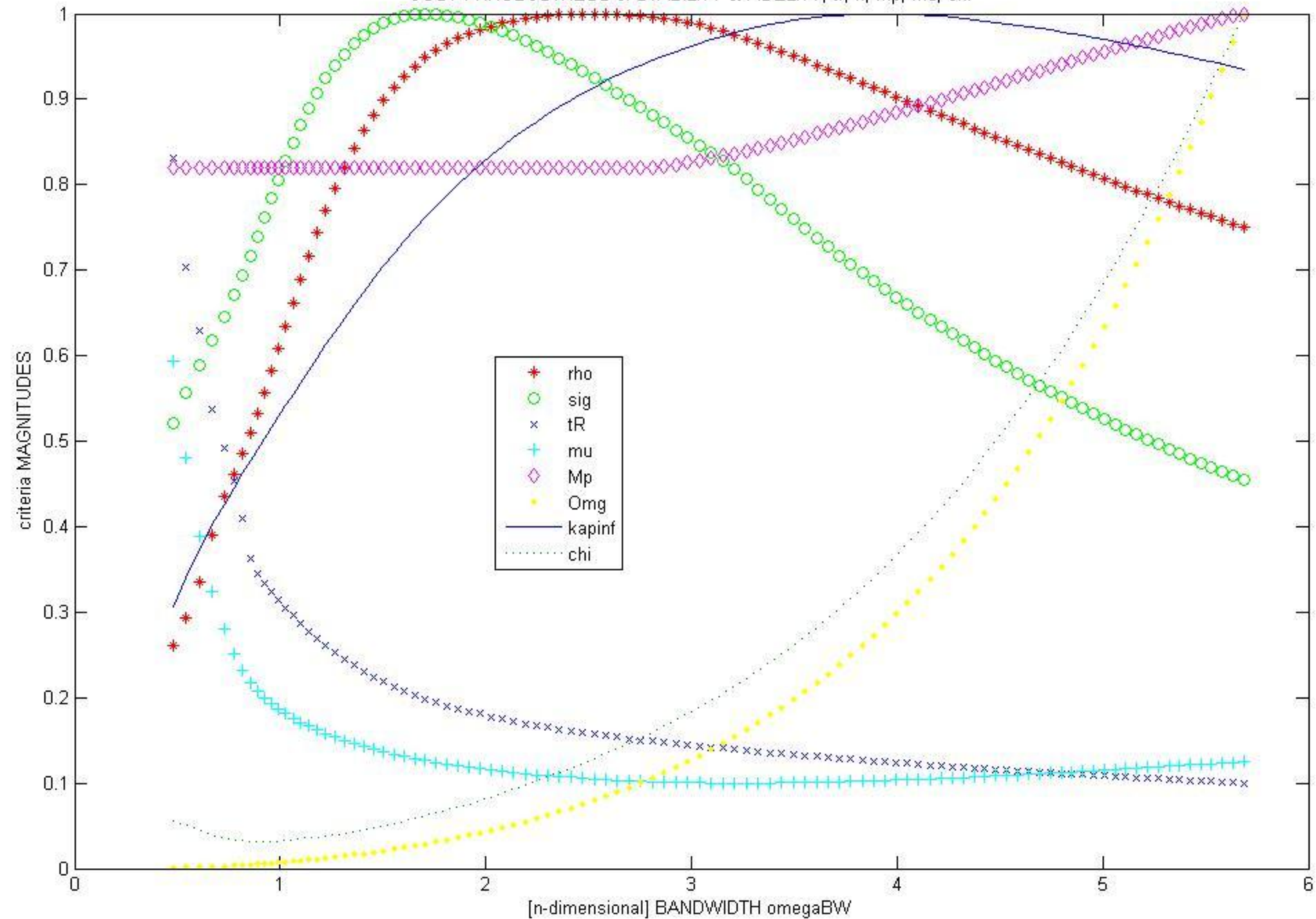
Also notice later that the QR design PRODUCES the clear possibility of resonant perturbations: $M_p > 1.25$!

[In these Figs, **all** curves are normalized to max = 1.]

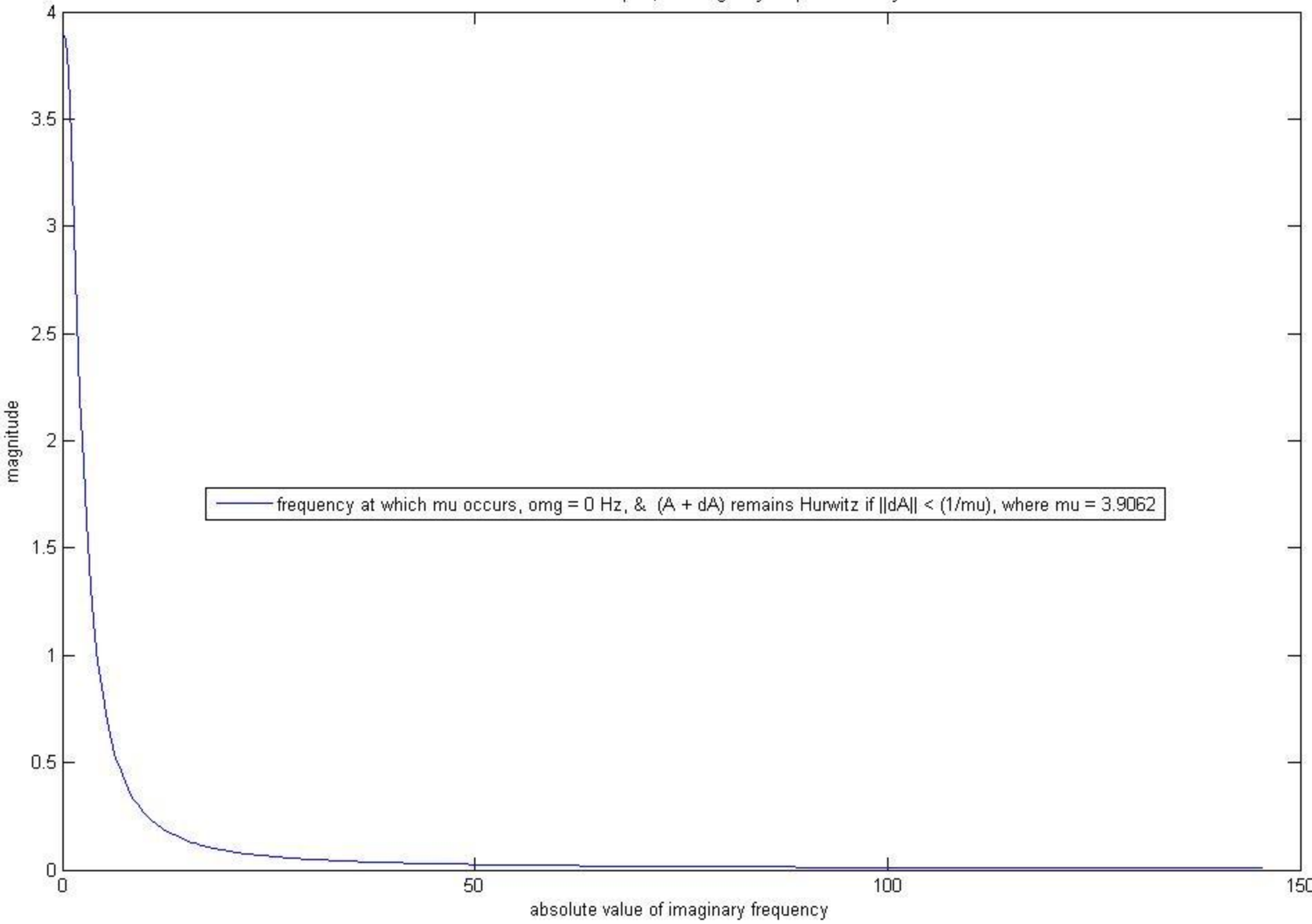
COST v RHOBUSTNESS of STABILITY & FIDELITY; tR; kapinf, Mp; mu; wBW; chi



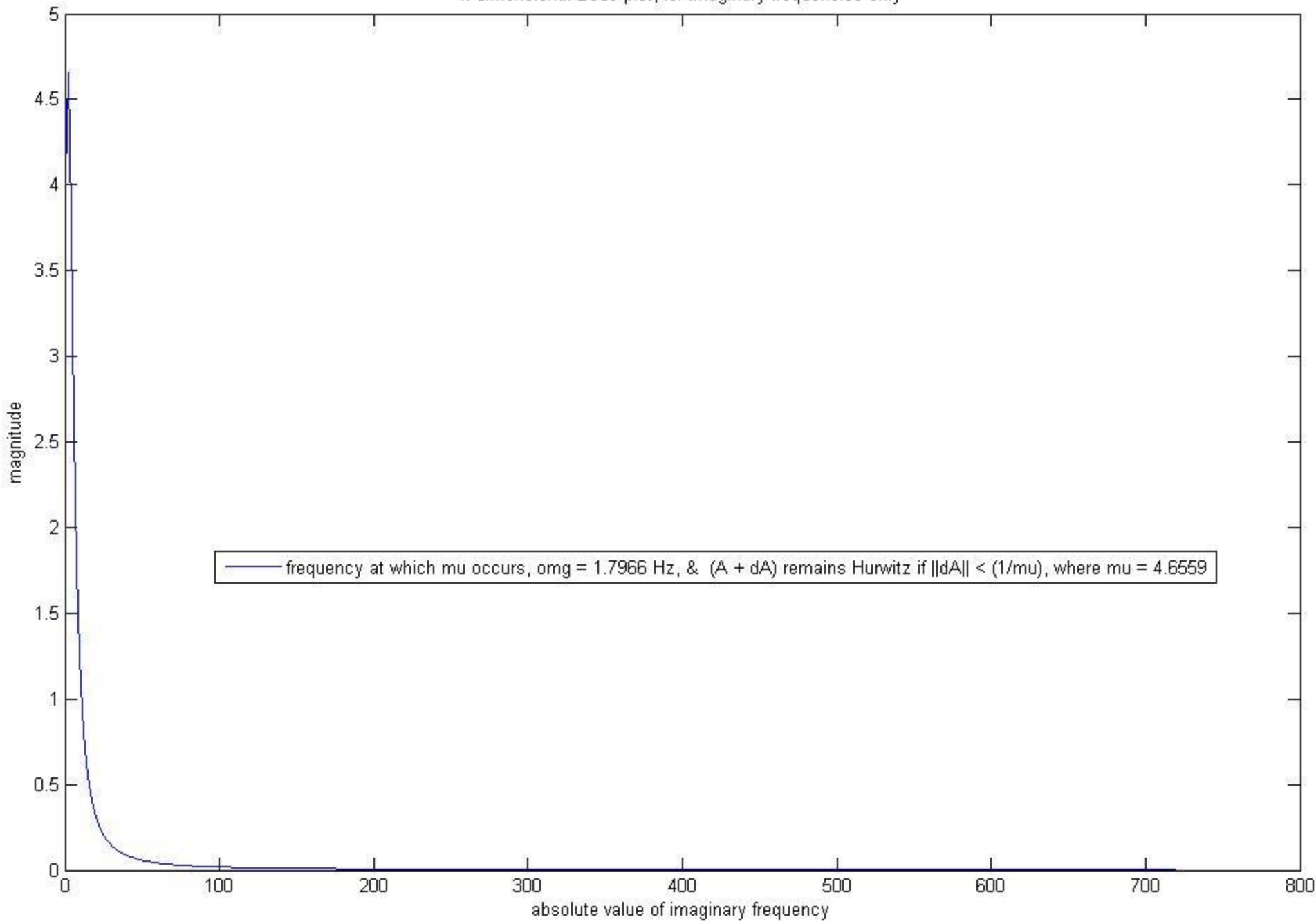
COST v RHOBUSTNESS of STABILITY & FIDELITY; tr, k; Mp, mu; chi



n-dimensional Bode plot, for imaginary frequencies only



n-dimensional Bode plot, for imaginary frequencies only





Krh_{is} =

0.9583	0.4168	-0.3663	-0.8765	0.1485	0.0030	-0.0552	0.1223
1.7074	-2.8704	0.1581	0.0727	1.4660	-0.3012	0.1567	-0.4016
-0.5996	0.2110	0.0019	0.2657	1.1320	0.0244	0.1039	0.8933
0.6885	-1.1104	0.0473	0.0119	0.5806	-0.1248	0.0829	-0.0936

whence Aclrh =

-0.1446	0.0347	0.0412	-0.2470	-0.0391	0.0053	-0.0062	0.0004
1.2158	-2.4510	0.1158	0.0424	1.0332	-0.2069	0.1156	-0.2788
9.9754	0.0539	-4.3502	-6.5277	1.6503	-0.1629	0.0619	0.2691
0	0	0.9986	0	0	0.0532	0	0
-0.0526	0.1217	-0.0134	-0.0117	-0.1827	0.0195	-0.0303	0.2872
-0.4560	1.2511	-0.2841	-0.4795	-1.5137	-1.2567	-0.6136	-0.4157
4.3812	-1.0754	-0.6277	-1.8341	-20.6907	-0.3396	-6.2697	11.2428
0	0	-0.0015	0	0	0.0289	1.0000	0

which produces the following closed-loop poles:



$$F_{clrh} = F - G.Krh, \text{ eig}F_{clrh} = \text{eig}(F_{clrh}) =$$

=

-3.9606

-2.3138

-1.2392 + 0.5091i

-1.2392 - 0.5091i

-1.3559 + 0.2638i

-1.3559 - 0.2638i

-1.5952 + 0.0732i

-1.5952 - 0.0732i

RhoSyn



help rhobassf

use:

[rho,lambd,gamm,tstr,tR,sig,kapinf,omgBW,Mp,Omg] = rhobassf(A,N);

[f denotes finite N-search]

inputs:

A = dynamical coefficient matrix of a stable LTI system
N = number of time-intervals over which to search, after
a preliminary estimate is obtained a la Wiberg

outputs:

rho = lambd/gamm = stability robustness margin, or "rhobustness"
lambd = reciprocal of e-folding response time tR
gamm = overshoot-factor constant
tstr = time t at which $y(t) = e.t.||\exp(A.t)||$ attains its maximum ymx
at which $\text{lambd} = 1/\text{tstr}$, $\text{gamm} = \text{ymax}/\text{tstr}$, $\text{rho} = 1/\text{ymx}$
tR = response time = $1/\text{lambd}$
sig = disturbance-variance rejection factor = $[(2/\text{lambd})^{1/2}].\text{rho}$
kapinf = accuracy or reciprocal stiffness or static loop gain = $1/||A||$
omgBW = bandwidth
Mp = peak resonance magnification factor
Omg = integrated power requirement or cost of control

These "Bass Characteristics" were presented at a US Army
Conference in 1983, after Bass at a 1956 BPS Proceedings,
and further elucidated in a 1991 CDC paper on RhoSyn



$$\left[\text{bsschrsForFClqr} \quad \text{bsschrsForFClrh} \quad \text{bsschrsForFClrh} / \text{bsschrsForFClqr} \right] =$$

=

rho	0.1099	0.1389	1.2639
sig	0.3927	0.4437	1.1296
trsp	4.3548	3.9002	0.8956
mu	4.5449	3.9062	0.8595
Mpeakres	1.3280	1.0000	0.7530
omegaBW	2.8429	2.5264	0.8887
Omega	5.2163	5.2344	1.0035
kappinf	0.2922	0.2560	0.8762
omegstr	0.8061	0	0
lambd	0.4087	0.4729	1.1569
gamm	3.7201	3.4051	0.9153
tstr	2.4466	2.1148	0.8644
tstrsg	2.1954	1.6208	0.7383

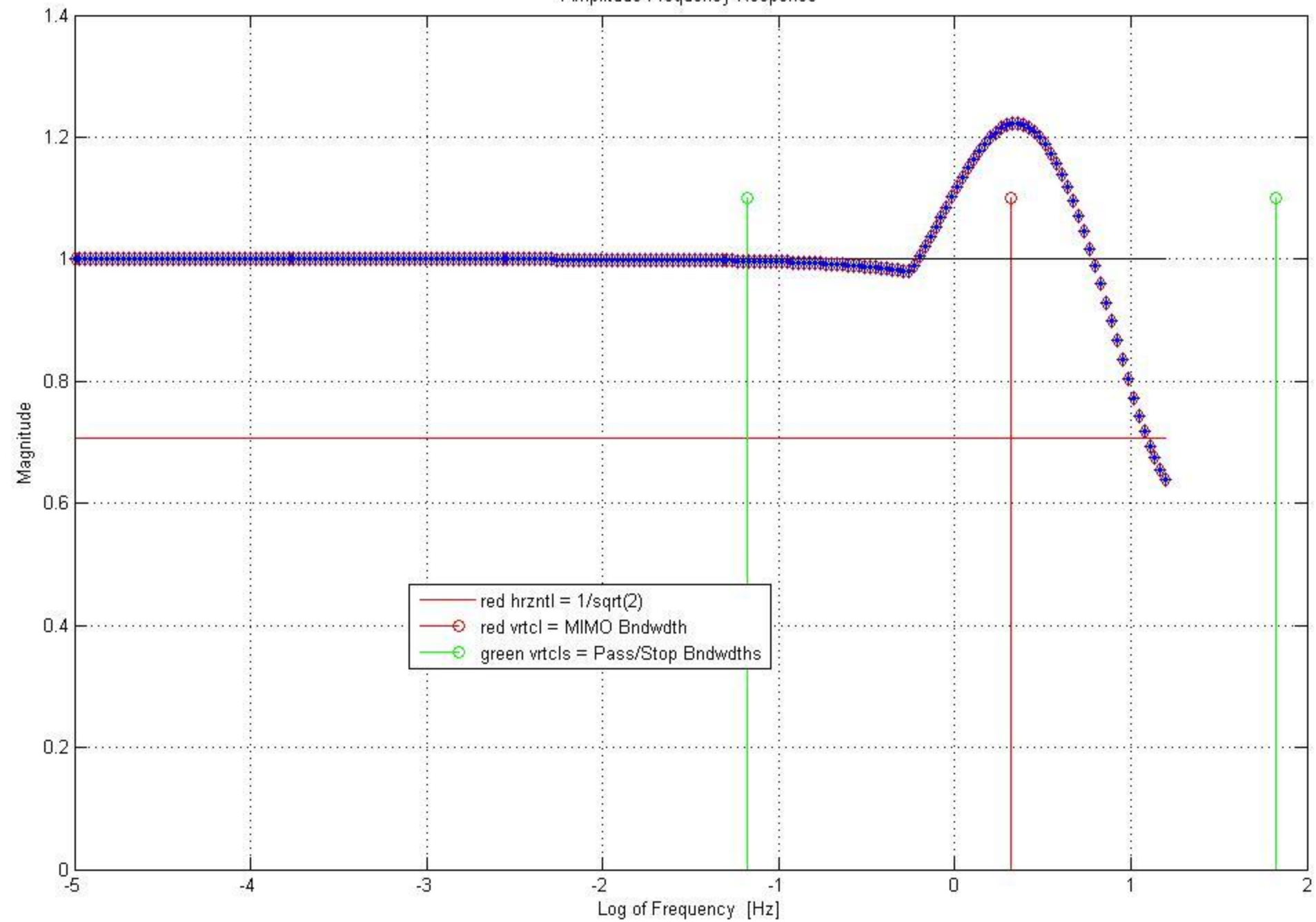


- RH design has **26% more stability robustness** than QR design
- RH design has **13% more fidelity robustness** than QR design
- RH design has **11.7% faster response** time than QR design
- RH design has **11.7% smaller mu** than QR design
- RH design has **Peak Resonance only 75%** that of QR design
- RH design requires **only 89% of Bandwidth** of QR design
- RH design requires **only 0.35% greater control energy** use as that of QR design
- RH design is only **88% as stiff** or accurate or has 88% **as much** static loop-gain as has that of QR design
- RH design has **15.7% larger damping factor** or error-decay exponent than QR design
- RH design has **8.47% smaller overshoot factor** than QR design

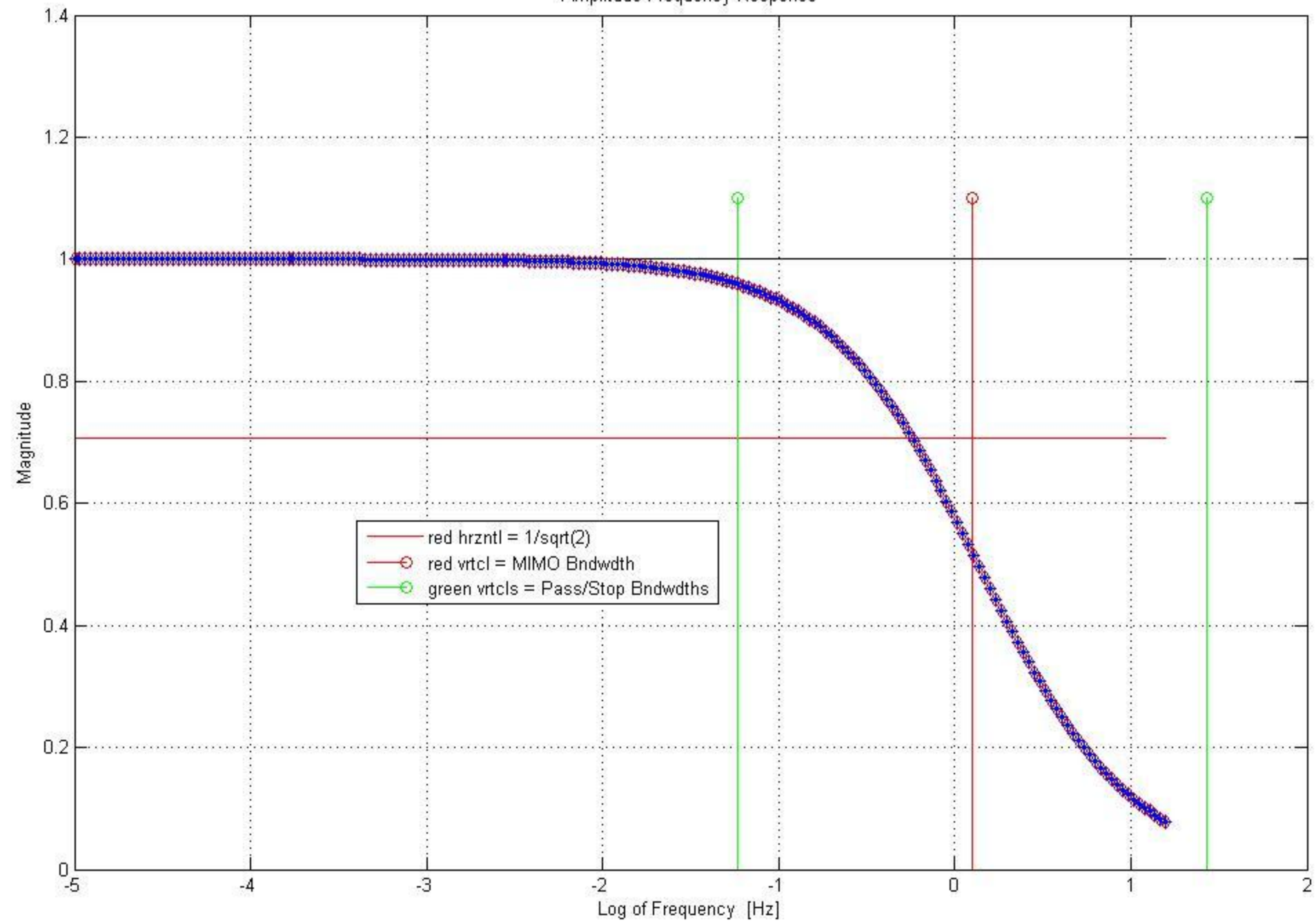


The next two slides
compare the MIMO
bandwidth of the **OH-6A**
SAS when designed by
LQR and when designed
by **RhoSyn**.

Amplitude Frequency Response



Amplitude Frequency Response





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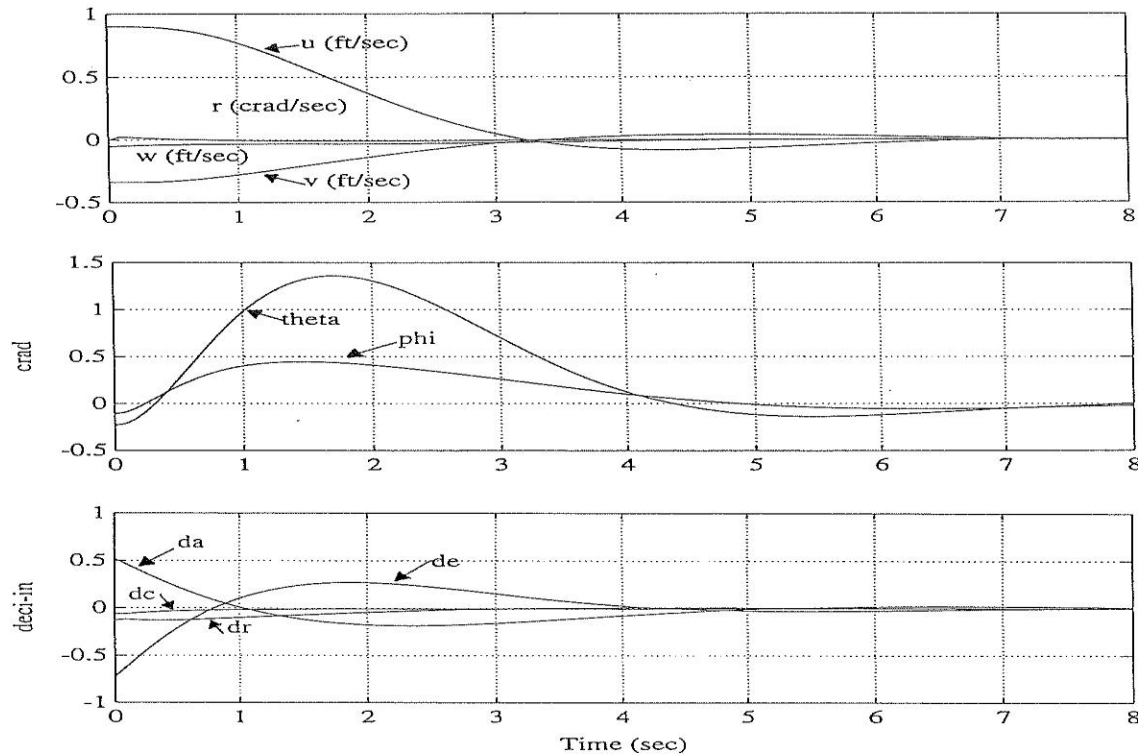
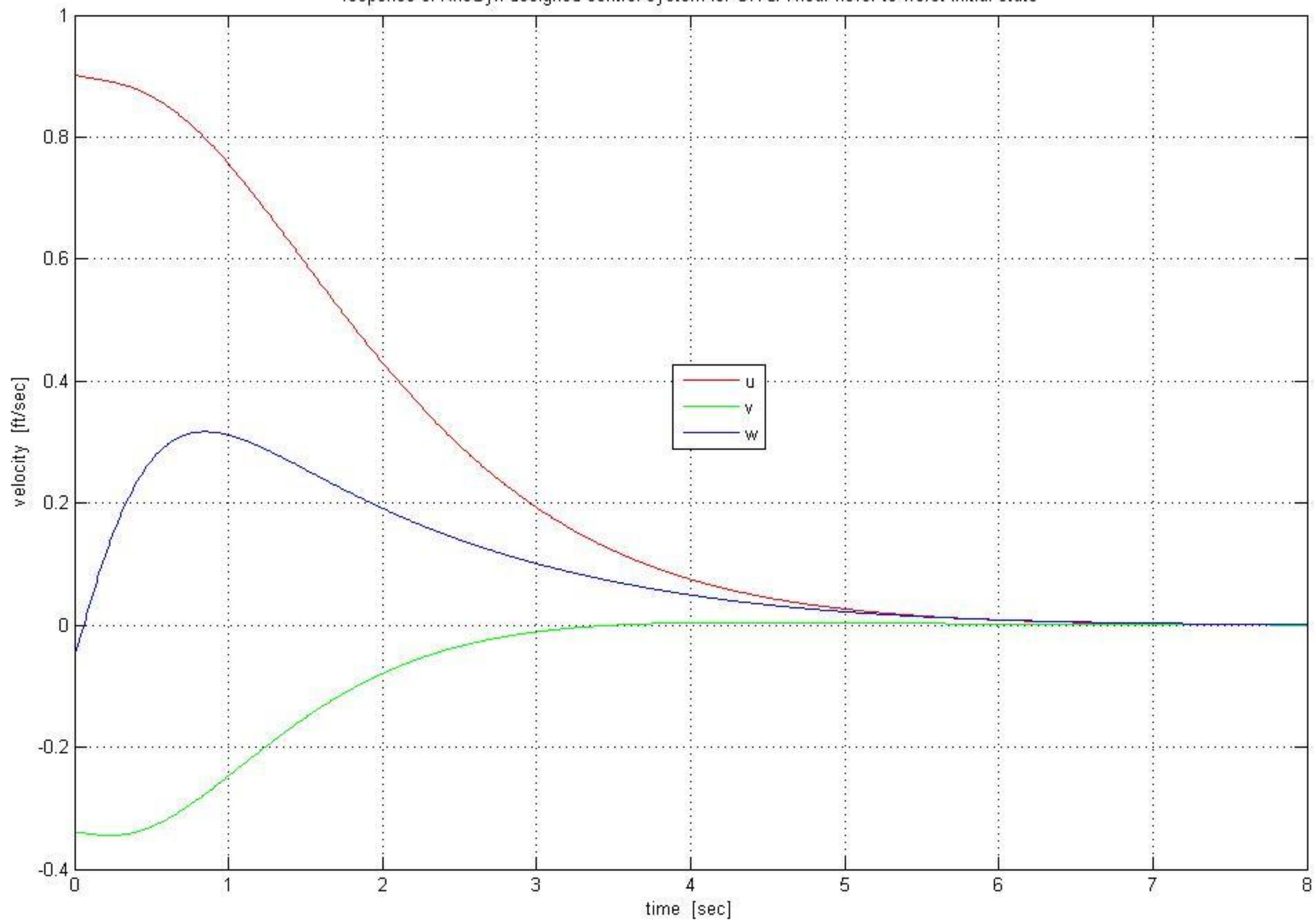


Figure 14.2. Response of the OH-6A helicopter near hover to the worst initial condition vector of unit magnitude with LQR SAS.

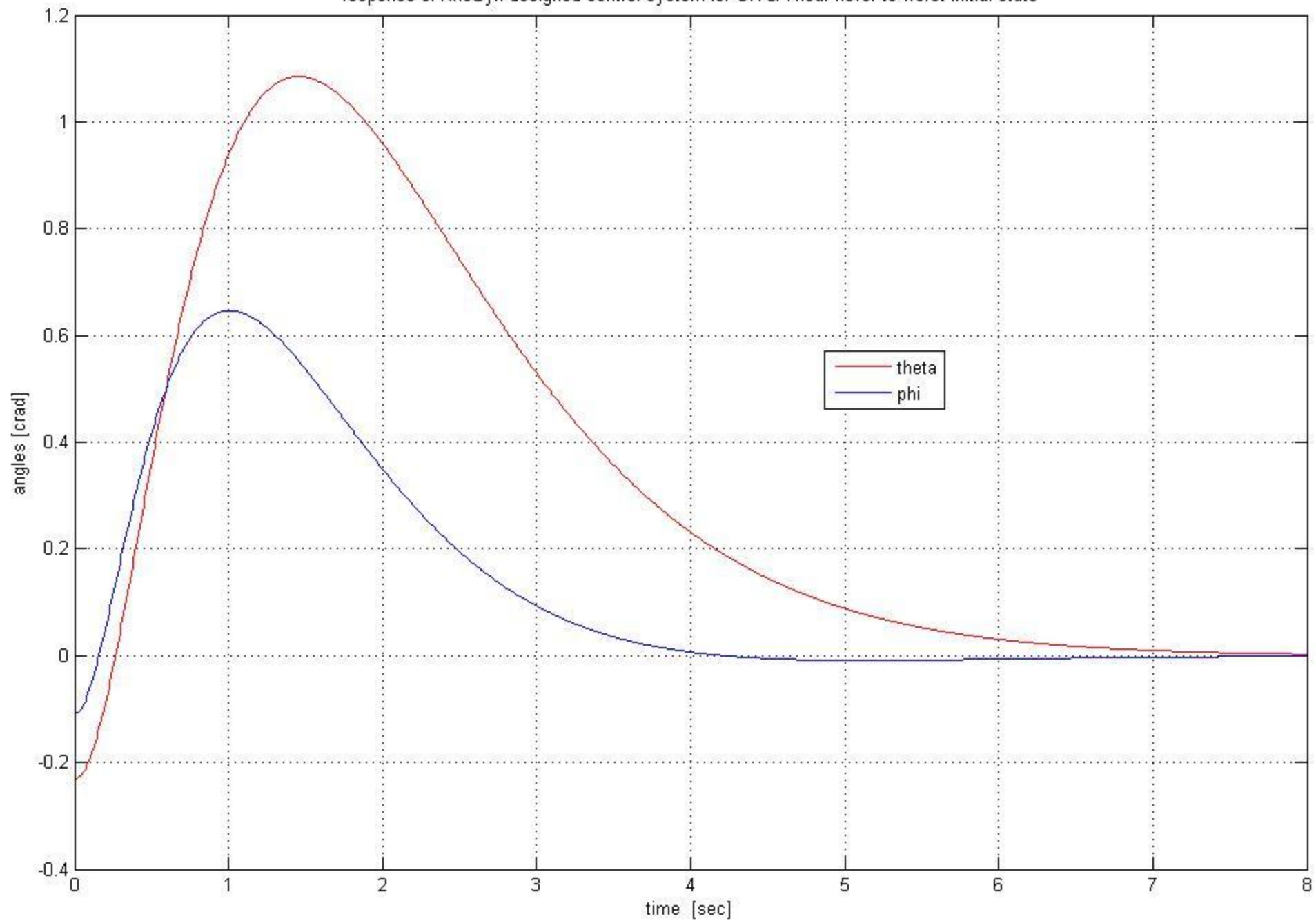
Problem 14.4.1 – *Response of the OH-6A to a Lateral Velocity and Yaw Rate Commands*

- Calculate and plot the response to a sideslip command, $v_c = 10$ ft/sec, while holding forward velocity, vertical velocity, and yaw rate equal to zero.
- Calculate and plot the response to a yaw rate command, $r_c = 1$ crad/sec, while holding forward velocity, vertical velocity, and lateral velocity equal to zero.

response of RhoSyn-designed control system for OH-6A near hover to worst initial state



response of RhoSyn-designed control system for OH-6A near hover to worst initial state





CONCLUSION

Although the LQR approach does an excellent job, the RhoSyn approach is **SIGNIFICANTLY** better!

Whenever a control system design problem can be formulated as an LQR-amenable problem, then a RhoSyn design should also be performed and the above-type of comparison should be made.

It should have been mentioned more explicitly above that in the crude “**NORM-BOUNDED** perturbation” approach used here, K_m is indeed identical to the reciprocal of μ when μ is defined *a la* Hewer *et al* as done hereinabove. But a wise designer will augment any RhoSyn design by complete MuSyn & KmSyn *comparative* designs.



Historical Acknowledgments:

Bass based his 1983 RhoSyn paper on direct first-hand knowledge of the 1960 work of Kalman on LQR Design Synthesis as it was originally done at Martin Marietta's RIAS, to which Bass had recruited Kalman in 1958, wherein Kalman had initially incorporated the cross-product terms, i.e. as a problem defined by (Q,R,S) instead of just merely (Q,R) alone.

Bass then worked on other subjects, such as the first derivation of the Kalman-Bucy Filter via Wiener's original method of Spectral Factorization, but had an opportunity to return to RhoSynthesis briefly in 1991 while at the Rockwell Science Center, but had not noticed that in 1986 Prof. Robert Stengel of Princeton University had published a book on *Stochastic Optimal Control* which **equally strongly** emphasized the importance of not assuming $S = 0!!$ In fact, RhoSyn can be regarded as a very slight modification of the *Implicit Model Following* approach to LQR design exposted on pages 434-438 of Stengel's *magnum opus*, ***Flight Dynamics***, Princeton University Press, 2004.



