Is The Albert Betz Law Stating The Maximum Wind Turbine Efficiency Of 16/27's Accurate?

Correcting The Betz Limit With Better Momentum Flux Theory

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Abstract

The more kinetic energy a wind turbine pulls out of the wind, the more the wind will be slowed down as it leaves the other side of the wind turbine. The core challenge is to maximize the efficiency of converting the drop in wind speed and pressure into useful mechanical work. The Betz law specifically claims that less than 16/27 (or 59.26%) of the kinetic energy in the wind can be converted to mechanical energy. The Betz limit is based upon maximizing a ratio of polynomials in the *Tip Speed Ratio* (λ) and concludes that the best that can be done, based on a rather crude *momentum-flux* theory, is to reduce the ambient wind speed by 1/3, so that the exit speed is 2/3 of the arriving speed. This paper uses a more sophisticated *momentum-flux* theory which says that you can reduce the exit speed by 1/2 of the arriving speed. This yields Cp efficiencies up to 97% instead of the $<$ 59.26% of the Betz theory. This error by Albert Betz in 1919 which has been widely adopted in the professional wind turbine literature [5] is exposed and rectified.

This paper resolves the long-standing Betz limit through a careful mathematical analysis of the conceptual wind turbine design, culminating in a Synthesis Theorem which proves that all salient wind turbine characteristics (including *Rotor Radius* (R), as well as Cp), are, in principle, uniquely determined, once the wind turbines rotor's *Tip Speed* (V_T), the rotor's mean blade-element profile *L*/*D* characteristics (C_l, C_d), and the rotor's geometric *Solidity* (σ) are specified, along with, in the Sunflower case (Figure 1), the *Aspect Ratio* ($0 < \xi < 1$), defined as the ratio of inner to outer radius of the cylindrical annular housing of two *non-axially-pivoted*, stacked, counter-rotating ring-blades, which have zero net angular momentum after being rotated by Grayson's patented [3] gear mechanism. The traditional HAWT (Horizontal Axis Wind Turbine) case (Figure 2) is recovered by setting $\xi = 0$.

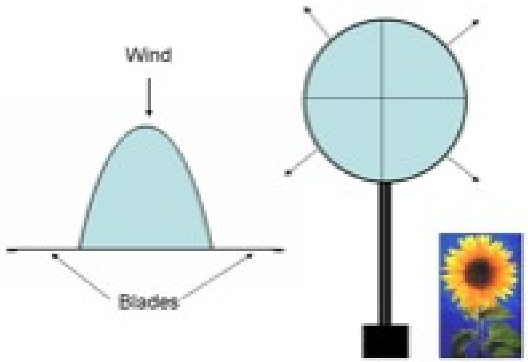

Fig. 1. HAWT with center parabolic diversion to counter-rotating ring-blades at the edge–Sunflower Design

Fig. 2. Traditional HAWT (Horizontal Axis Wind Turbine)

I. NOMENCLATURE

V_T	rotor (blade) tip-speed
V_e	wind exit velocity
V_W	arriving wind speed
λ	tip speed ratio
C_l	mean blade-element lift-coefficient
C_d	mean blade-element drag-coefficient
ρ	ambient air density
R	rotor radius
σ	rotor solidity (ratio of blade-area to aperature-area)
P_i	arriving power
P_p	principal energy loss of rotor blade
P	net power retained
ε	drag power/lifting power
Cp	rotor efficiency (power coefficient of the rotor)
$D.L.$	disk loading
$T.L.$	thrust loading
$L.L.$	lift loading
ξ	aspect ratio
η	lifting efficiency
W_G	gross weight

II. INTRODUCTION

Assume a HAWT with a turbine rotor of radius R, solidity σ, tip-speed V_T, and mean blade-element drag coefficient C_d, operating in air of ambient density ρ. Given that the *arriving power* P_i required for lift and the *principal energy loss* P_p wasted in overcoming rotor drag are given by

$$\begin{aligned} P_i &= \frac{\sqrt{W_G^3}}{R\sqrt{2\pi\rho}\sqrt{1-\xi^2}} && (1) \\ P_p &= \frac{1}{8}(\sigma C_d)(\pi(1+\xi)(1-\xi^4)R^2)(\rho V_T^3) && (2) \end{aligned}$$

the *total arriving power*, P, is given approximately by

$$P = P_i + P_p \tag{3}$$

Accordingly, a **physically significant** measure of the *rotor's lift efficiency* is given by the Wind Turbine Efficiency, Cp, which is defined as the ratio of arriving power to total power

$$\begin{aligned} Cp &= \frac{P_i}{P} \\ &= \frac{1}{1+\varepsilon} && (4) \end{aligned}$$

where $\varepsilon = \frac{P_p}{P_i}$. It is well known (and proved below) that ε, hence Cp, is a constant independent of (ρ, V_T, R) and dependent only upon the rotor's *geometry* and *aerodynamic* characteristics (σ, ξ, C_l, C_d); where C_l is the rotor blade-element's mean lift coefficient and C_d is its mean drag coefficient. In fact, it can be proved (as in e.g. [3],[5]-[7]) (and re-derived below in the case $\xi > 0$) that

$$\varepsilon = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{\sigma}\left(\frac{\sqrt{C_l^3}}{C_d}\right)}\left(\frac{(1+\xi)(1+\xi^2)}{\sqrt{(1+\xi+\xi^2)^3}}\right). \tag{5}$$

As stated in the early literature ([1], [7]), in the case where $\xi = 0$, the total arriving power, P, can be *minimized* by appropriate choice of radius R, and for the optimum radius, $P_i = 2P_p$ (i.e. allegedly **optimum** ε = 1/2), Cp for the **optimum** physical rotor is equal to 2/3 [7].

Despite an apparently convincing mathematical "proof" of this claimed (and, if true, obviously basic) optimization result, its irrelevance to engineering reality has been recognized since at least 1967 [7]. According to McCormick [6], the "foregoing analysis is possibly not too realistic...In several respects Cp is useless. As an academic exercise, it is interesting, but in practical applications we must consider the power. McCormick also states that ”a typical Cp = 0.76.”

III. RESOLUTION OF BETZ THEORY

So what is wrong with the accepted proof that ”optimum Cp = 16/27 or 59.29%”?

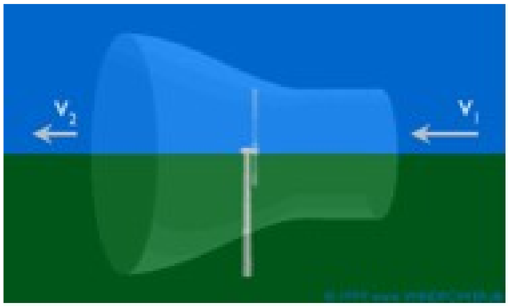

Fig. 3. Wind speed before the wind turbine (v_1) and after the passage through the rotor plane (v_2)

Let us make the reasonable assumption that the average wind speed through the rotor area is the average of the undisturbed wind speed before the wind turbine, v_1, and the wind speed after the passage through the rotor plane, v_2, i.e. (v_1 + v_2)/2. The mass of the air streaming through the rotor during one second is

$$m = \rho F \left(\frac{v_1 + v_2}{2} \right) \tag{6}$$

where m is the mass per second, ρ is the density of air, F is the swept rotor area and $(v_1 + v_2)/2$ is the average wind speed through the rotor area. The power extracted from the wind by the rotor is equal to the mass times the drop in the wind speed squared (according to Newton's second law)

$$P = \frac{1}{2} m \left(v_1^2 - v_2^2 \right) . \tag{7}$$

Substituting Equation 6 into Equation 7 we get the following expression for the power extracted from the wind

$$P = \frac{\rho F}{4} (v_1 + v_2) \left(v_1^2 - v_2^2 \right) . \tag{8}$$

Now, let us compare our result with the total power P_0, in the undisturbed wind streaming through exactly the same area F, with no rotor blocking the wind

$$P_0 = \frac{\rho}{2} v_1^3 F. \tag{9}$$

The ratio between the power we extract from the wind and the power in the undisturbed wind is then

$$\frac{P}{P_0} = \frac{1}{2}\left(1 - \left(\frac{v_2}{v_1}\right)^2\right)\left(1 + \left(\frac{v_2}{v_1}\right)\right) \tag{10}$$

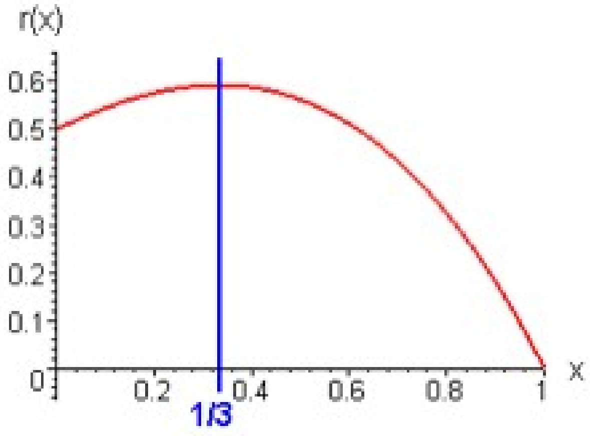

Fig. 4. Ratio between the power extracted from the wind and the power in the undisturbed wind (P/P_0) as a function of v_2/v_1

As seen in Figure 4 the maximum amount of power that is extracted from the wind is 0.5929 or 16/27 of the total power in the wind and occurs at v_2/v_1 = 1/3.

The fallacy in the preceding long-overlooked error in the wind literature was discovered as a result of the careful formulation and proof of the theorem in the following section, pertaining to preliminary conceptual VTOL (Vertical Take Off and Landing) rotorcraft designs [4] and then adapted to wind turbine HAWT designs. The theorem uses the conservation of mass, momentum and energy.

IV. WIND TURBINE SYNTHESIS THEOREM (BASS LAW)

In order to proceed, we must recall the additional concepts of Disk Loading $D.L.$, Thrust Loading $T.L.$, Lift Loading $L.L.$ and wind turbine exit velocity V_W as defined in the Nomenclature above.

Theorem 1 (Synthesis Theorem): Given the basic design parameters (W_G, ρ, σ, ξ, C_l, C_d, V_T) the remaining design and performance characteristics (ε, Cp, V_w, $D.L.$, $L.L.$, P, R) are uniquely determined and in fact given by the following formulae.

$$\begin{aligned}
\varepsilon &= \left(\frac{3\sqrt{3}}{2}\right)\left(\sqrt{\sigma}\left(\frac{\sqrt{C_l^3}}{C_d}\right)\right) && (11)\\
\frac{1}{2}V_W &= \left(\frac{1}{2\sqrt{3}}\right)\left(\sqrt{\sigma C_l}\right)V_T && (12)\\
\lambda &= \frac{V_T}{V_e} = \frac{B}{\sqrt{\sigma}}\\
B &= \sqrt{\frac{3}{C_l\left(1+\xi+\xi^2\right)}}\\
Cp &= \frac{P}{P_p} = 1-\varepsilon = 1-\frac{\lambda}{\lambda_{\text{max}}} && (13)\\
\lambda_{\text{max}} &= \frac{2}{3}\frac{C_l}{C_d}\frac{1+\xi+\xi^2}{1+\xi+\xi^2+\xi^3} && (14)\\
D.L. &= \frac{T.L.}{\sigma} && (15)\\
T.L. &= 2\rho V_W^2 = \frac{1}{3}(\sigma C_l)q_T\\
q_T &= \frac{1}{2}\rho V_T^2\\
L.L &= \frac{Cp}{V_W} && (16)\\
P &= \frac{1}{Cp}W_G V_W && (17)\\
R &= \left(\sqrt{\frac{W_G}{2\pi\rho}}\right)\frac{1}{V_W} && (18)
\end{aligned}$$

Proof: The first item is to establish the basic formula for *arriving power* P_i, which is a consequence of *conservation* of **mass**, **momentum** and **energy** (using the standard notation that $\dot{m}$ = dm/dt). Conservation of mass and mass-flux states that

$$\begin{aligned}
\dot{m} &= \rho V_W A\\
&\equiv \rho V_W A_W && (19)
\end{aligned}$$

where $A = \pi\left(1-\xi^2\right)R^2$ is the area through which ambient air of density ρ flows through the wind turbine with exit velocity V_W, and where (V_W, A_W) are the corresponding [unknown!] final (remote) wake velocity and wake cross-sectional area far below the rotor. Conservation of mass-flux is envisioned in a flux-tube bounded by streamlines analogous to the generators of a curvilinear cylindrical annulus, whose top and bottom areas are respectively A and A_W. Next, Newton's First Law is recalled in the form that Force F, or Thrust T, (in this case, equal to gross weight W_G) is equal to the rate of change of momentum (momentum flux) according to the conservation of momentum.

$$\begin{aligned} F &= T = W_G \\ &= (\dot{m}V)_{\text{wake}} - (\dot{m}V)_{\text{air above rotor}} \\ &\equiv \dot{m}(V_W - 0) \\ &\equiv \dot{m}V_W \\ &= (\rho V_W A)\, V_W \end{aligned} \tag{20}$$

Finally, recall that power is the rate at which a force does work according to conservation of energy.

$$\begin{aligned} P &= \frac{1}{2}\dot{m}V_W^2 \\ &= \frac{1}{2}(\rho V_W A)\, V_W^2 \\ &= \frac{1}{2}F V_W \end{aligned} \tag{21}$$

From the two different versions of Equation 21, one finds that

$$V_e = \frac{1}{2}V_W \tag{22}$$

which, when inserted into Equation 20 along with the definition of A, yields

$$\begin{aligned} F &= T = W_G \\ &= (2\rho\pi(1-\xi^2)R^2)V_W^2 \end{aligned} \tag{23}$$

which, implies that

$$V_W = \sqrt{\frac{W_G}{2\rho\pi}}\frac{1}{\sqrt{1-\xi^2}R} \tag{24}$$

which, when inserted into the first version of Equation 21, yields

$$\begin{aligned} P_i &= 2\rho\pi(1-\xi^2)R^2V_W^3 \\ &= 2\rho\pi(1-\xi^2)R^2\left(\sqrt{\frac{W_G}{2\rho\pi}}\frac{1}{\sqrt{1-\xi^2}R}\right)^3 \\ &= \frac{\sqrt{W_G^3}}{(\sqrt{2\rho\pi})(\sqrt{1-\xi^2})R} \end{aligned} \tag{25}$$

for *arriving power* as claimed, thereby justifying Equation 1. The result (Equation 25) is called the *momentum-flux* theory.

Further progress requires use of the *blade-element theory*. Suppose that there are N_B blades, each of mean chord c and length $(1-\xi)R$. Let dr denote an element of length of one blade at station r, so that the area element $dA = cdr$. If the blade tip is moving with velocity V_T, then the velocity of the element dA is $V(r) = \frac{r}{R}V_T$ because all stations on the blade have the same *angular* velocity $\omega = \frac{V_T}{R}$ and $V(r) = \omega r$. Let $c_l(r)$ denote the *lift coefficient* at the station r; it is well known that c_l is a function of the (local) angle of attack, but we regard the collective pitch of the rotor blades as fixed and consider only the dependence upon r. It is also well known that the lift dT provided by the blade element dA is given by $dT = c_l(r)q(r)dA$, where at station r the *dynamic pressure*

$$\begin{aligned} q(r) &= \frac{\rho}{2}V(r)^2 \\ &= \frac{\rho}{2}\frac{r^2V_T^2}{R^2} \\ &= q_T\left(\frac{r}{R}\right)^2 \end{aligned} \tag{26}$$

where by definition

$$q_T = \frac{\rho}{2}V_T^2 \tag{27}$$

Now the total lift T is given by integrating over r and then multiplying by N_B, according to which

$$\begin{aligned} T &= W_G = N_B\int_{\xi_R}^{R} dT \\ &= N_B\int_{\xi_R}^{R} c_l(r)q(r)dA \\ &= N_B\int_{\xi_R}^{R} c_l(r)q(r)cdr \\ &= \frac{N_Bcq_T}{R^2}\int_{\xi_R}^{R} c_l(r)r^2dr \\ &= \frac{(\pi R(1+\xi)\sigma)q_TC_l}{R^2}\int_{\xi_R}^{R} r^2dr \\ &= \frac{1}{3}\sigma C_l(\pi(1+\xi)(1-\xi^3)R^2)q_T \end{aligned} \tag{28}$$

where in the penultimate step we have used the definition of mean as opposed to local lift coefficient, and have introduced the concept of *solidity*, σ, defined as the ratio of the rotor blades area, A_B, to the total disk area, A_R, namely

$$\begin{aligned}\sigma &= \frac{A_B}{A_R} = \frac{N_B(c(1-\xi)R}{A_R} \\ &= (N_B c)(1-\xi)\frac{R}{\pi(1-\xi^2)R^2} \\ &= \frac{N_B c}{\pi(1+\xi)R} \qquad (29)\end{aligned}$$

From Equations 27, 28 and 24 we now have two separate relationships for *Thrust Loading* $T.L.$, namely, using the identity $(1+\xi)(1-\xi^3) \equiv (1-\xi^2)(1+\xi+\xi^2)$,

$$\begin{aligned}T.L. &= \frac{W_G}{\pi(1-\xi^2)R^2} \\ &= \frac{1}{3}\sigma(1+\xi+\xi^2)C_l q_T \\ &= 2\rho V_W^2 \qquad (30)\end{aligned}$$

where the second version comes from Equation 23 after squaring and rearranging Equation 24. We can now compute the *arriving power* from the blade-element theory, namely from Equation 21, 28 and 30 solved for V_W, i.e.

$$\begin{aligned}P_i &= TV_W = W_G V_W \\ &= \frac{1}{3}\sigma(1+\xi+\xi^2)C_l q_T(\pi(1-\xi^2)R^2)\sqrt{\frac{\sigma(1+\xi+\xi^2)C_l q_T}{6\rho}} \\ &= \frac{1}{12\sqrt{3}}\sqrt{(\sigma(1+\xi+\xi^2)C_l)^3}(\pi(1-\xi^2)R^2)p_T \qquad (31)\end{aligned}$$

where

$$p_T = \rho V_T^3. \qquad (32)$$

We next need an expression for the principal energy loss, which is derived from the blade-element theory in a manner similar to that for the arriving power. The drag on a blade element at station r is given by $dD = c_d(r)q(r)dA$, so the element of power loss to this drag is $dP_p = V(r)dD$, where, as before,

$V(r) = \omega r = \frac{r}{R} V_T$. Accordingly, by Equation 29,

$$\begin{aligned} P_p &= N_B \int_{\xi_R}^{R} c_d(r) q(r) V(r) dA \\ &= \frac{N_B c}{2} \frac{\rho V_T^3}{R^3} \int_{\xi_R}^{R} c_d(r) r^3 dr \\ &= \frac{1}{2R^3} (\pi(1+\xi) R\sigma) p_T C_d \int_{\xi_R}^{R} r^3 dr \\ &= \frac{1}{8} \sigma C_d (\pi(1+\xi)(1-\xi^4) R^2) p_T. \end{aligned} \tag{33}$$

We have now proved the expression given in Equation 2, thus completing the proof of Equation 3.

Now use the identity $(1-\xi^4) \equiv (1-\xi^2)(1+\xi^2)$ and divide Equation 33 by Equation 31 to prove Equation 5. Since Equation 4 is just a definition, we have now verified Equations 3 - 5 as promised, and are ready to prove the remainder of the claimed theorem. Note that Equations 11 and 13 are just a rearrangement of Equations 5 and 4, so we turn to Equation 12, which, by Equation 27 is an easy consequence of rewriting Equation 30 in the form

$$2\rho V_W^2 = \frac{1}{6} C_l (1+\xi+\xi^2) \rho V_T^2 \tag{34}$$

Next, Equation 15 was already proved in Equation 30. Also, from the 2nd relationship in Equation 31, $C_p P \equiv P_i = W_G V_W$, which proves Equation 17. Then using Equation 17 and the definition of lift loading as $\frac{W_G}{P}$, Equation 16 follows immediately. Finally, Equation 18 is a corollary of Equation 24.

This completes the proof of Equations 11 - 18 and therefore proves the theorem. ■

V. CONCLUSION

The Cp analysis is critical in preliminary comparison of different possible HAWT designs which are identical except for the geometric & aerodynamic parameter set (σ, ξ, C_l, C_d). By the same kind of analysis as that detailed above, it is possible to further manipulate the preceding equations to re-state the principal results from a different perspective, namely that of pure Lifting Efficiency in terms of Gross Weight W_G versus required horsepower P. In particular, *Arriving power* P_i can be displayed as

$$P_i = \frac{1}{2\sqrt{3}} \sqrt{\sigma C_l} \sqrt{1+\xi+\xi^2} W_G V_T \tag{35}$$

while the corresponding *Profile Power* P_p can be displayed as

$$P_p = \frac{3}{4}\frac{C_d}{C_l}\left(1+\frac{\xi^3}{1+\xi+\xi^2}\right)W_G V_T \tag{36}$$

The reader may readily verify the validity of the preceding formulae upon computing Cp and verifying that the answer Equations 4 and 5 already carefully derived during the Proof of the Rotorcraft Synthesis Theorem is recovered. Now define *Lifitng Efficiency* η as Gross Weight W_G divided by the required total power $P(\xi) = P_i + P_p$ after multiplying the numerator by tip-speed V_T in order to make the ratio dimensionless:

$$\begin{aligned}
\eta^{-1} &= \frac{P(\xi)}{W_G V_T} \\
&= \frac{1}{2\cdot 3^{1/2}}(\sigma C_l)^{1/2}(1+\xi+\xi^2)^{1/2} + \frac{3}{4}\frac{C_d}{C_l}\left(1+\frac{\xi^3}{1+\xi+\xi^2}\right) \\
&\geq \frac{P(0)}{W_G V_T} \equiv \frac{1}{2\cdot 3^{1/2}}(\sigma C_l)^{1/2} + \frac{3}{4}\frac{C_d}{C_l} \\
&\geq \frac{4}{9}\frac{C_l}{C_d}
\end{aligned} \tag{37}$$

if $\sigma \geq \sigma_{min}$, where the final step in Equation 37, left to the reader, is an elementary exercise in differential calculus and curve minimization (regarding C_l as the only variable), which is feasible provided $\sigma \geq \sigma_{min} = 27\frac{C_d^2}{C_l^3}$. Since $\sigma_{\min}$ is almost always smaller than the actual σ, one could *normalize* η to be less than unity by dividing it by $\frac{4}{9}\frac{C_l}{C_d}$ and calling the result $\eta_n = \frac{9}{4}\frac{C_d}{C_l}\eta$, and then replace Equation 37 by

$$\eta_n(\xi) \leq \eta_n(0) \leq 1 \tag{38}$$

The significance of Equation 38 is that

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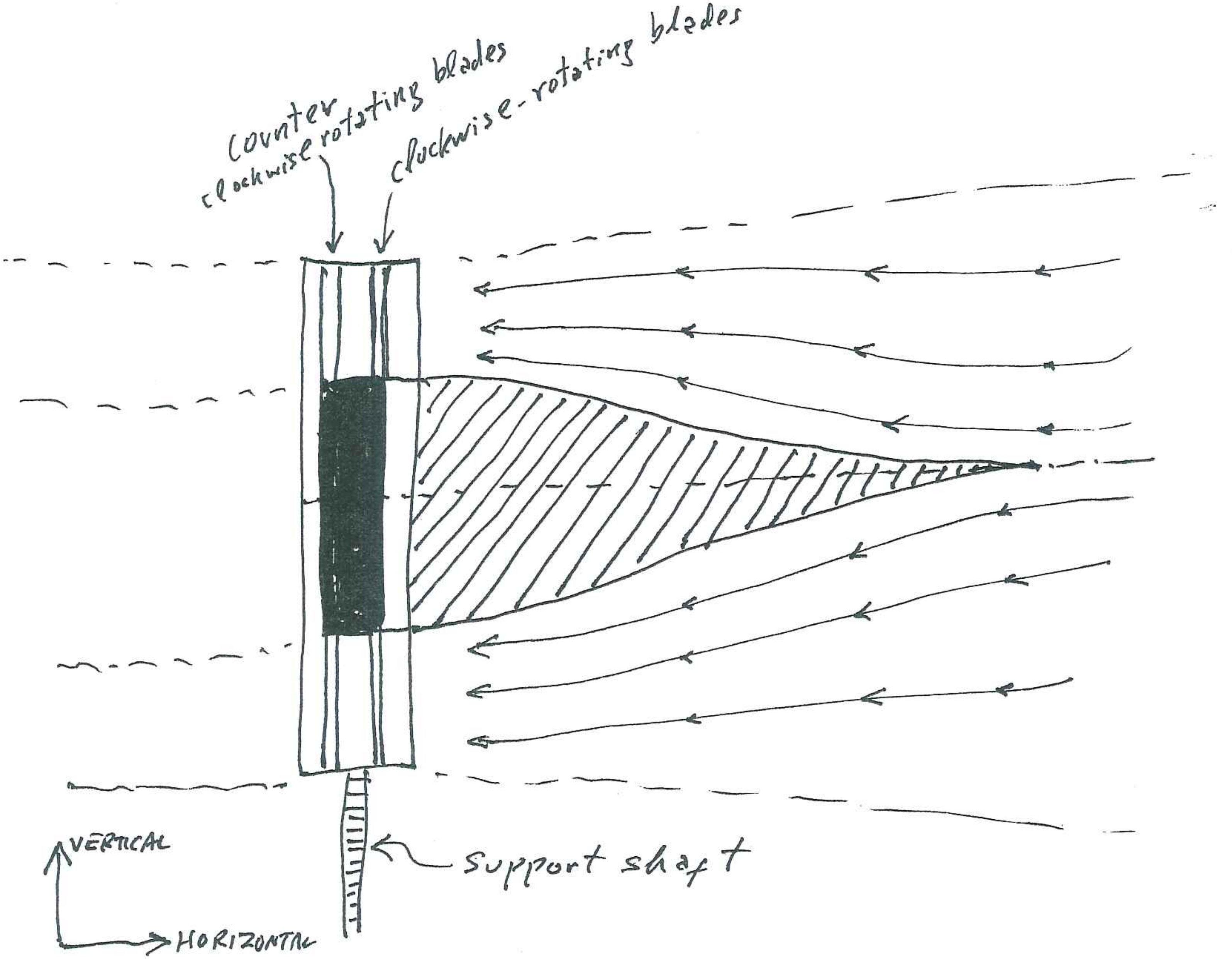
Counter clockwise rotating blades
clockwise-rotating blades
support shaft
VERTICAL
HORIZONTAL