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Brief Summary in Historical Context

The appended paper by R. W. Bass, if it contains no further fatal mathematical errors besides one noted in 2003 by Gordon Emslie and now apparently fixed, purports to be the most significant advance, during the *past 317 years*, in the most celebrated problem in mathematics, and has as a corollary an immediate resolution of the chief remaining unresolved puzzle in Dynamical Astronomy for the *past 237 years*.

To confirm these statements, consult K.R. Meyers' book on the *N-Body Problem* (Springer-Verlag, 1992) and verify that on page 18 he refers to the *Newtonian 3-Body Problem* (**1686**) as **"the most celebrated problem in mathematics."**

Also, as every history of dynamical astronomy recalls, the empirical fact that each planet is roughly twice as far from the Sun as its predecessor has been known since Titius discovered it in **1766** and Bode popularized the observation in **1778** (later acknowledging Titius' priority); moreover, this regularity was famously used in both the *discovery* of Uranus (**1781**) and in the *discovery* of the Asteroid Belt (**1801**).

In his 1993 book (cited in the paper), former Naval Observatory dynamical astronomer Dr. Tom Van Flandern cites multiple, convergent, astrophysical evidences not easily explained except by the hypothesis that a planet once occupied the Asteroid Belt and for some reason dissipated explosively in the geologically not far distant past. Whether this is correct or not, the late astronomer and *FRAS*, Dr. Michael Ovenden, found that the present resonant dynamical structure of the solar system makes most sense [his **1972** *Principle of Least Interaction Action*, which he later graciously noted in print had been anticipated by R.W. Bass in a preprint distributed at the International Congress of Mathematicians in Edinburgh (**1958**) and in a paper presented to the IAF in Stockholm by Bass in **1960**] only if the Asteroid Belt's location is regarded as that of a former planet. In such a case, one would count 11 known primary bodies in the solar system, including the Sun.

In **1992**, Dr. C.D. Johnson *et al* at UAH did a "naturally-weighted" alternative to the standard Gaussian Least Squares *numerical fit* to the *mean* planetary distal ratio β for **all** 11 bodies at once, arriving at the collective answer of

(**EMPIRICAL** 1992) $\beta = 1.795$.

More recently, Chapman [*JRASC*, 2001] and Lynch [*MNRAS*, 2003] have simply averaged the 9 actual distal ratios for the 10 "bodies," including the asteroid Ceres, finding

(**EMPIRICAL** 2001) $\beta = 1.710 \pm 0.0300$,
(**EMPIRICAL** 2003) $\beta = 1.706 \pm 0.0544$.

In **1995**, while involuntarily unemployed (except for honoraria from David Talbott, former publisher of *Pensee*) Bass revisited his **1958** work and, asking what ratio would **not** cause the inner body to **resonantly de-stabilize** the outer body (or conversely), derived rigorously several **new natural constants** (like Napier's *e*, or Euler's constant *c*), which are **independent** of the ratios of the masses of the planets to that of the Sun, or even of the Newton-Cavendish gravitational constant *G*, namely [pending confirmation by more powerful computers] there are only **FIVE** possible **truly orbitally stable** solutions of the Newtonian 3-Body Problem near a *Copernican-Kepler configuration*, which must be associated with **resonances** (otherwise, by the Newcomb/Poincare/KAM theorem there are **chaotic** motions arbitrarily nearby) and are:

(**THEORETICAL** 1995/2003) resonance: (**2,1**), $\beta = 1.70716761693332$,
(**THEORETICAL** 1995/2003) resonance: (**3,1**), $\beta = 1.77317834465894$,
(**THEORETICAL** 1995/2003) resonance: (**4,1**), $\beta = 1.80915773588908$,
(**THEORETICAL** 1995/2003) resonance: (**3,2**), $\beta = 2.70842603143354$,
(**THEORETICAL** 1995/2003) resonance: (**5,2**), $\beta = 1.77256086302991$.

The author will supply MATLAB programs to anyone who would like to verify that the numerical search method is both valid and exhaustive.

CORRIGENDA TO

DYNAMICAL DERIVATION OF BODE'S LAW

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Instructions for “fixing” the only error in the paper as printed and verifying the newly corrected Abstract.

This is an edited excerpt from a personal letter, sent 6/25/2006 to Leroy Ellenberger in reply to his query as to why I had not placed this paper on my website www.innoventek.com in its Science sub-site.

About 99% of the paper is OK, but there is one bad mistake in it, which is correctible but I haven't had time to do it explicitly in a nicely-formatted mathematically-typeset version yet.

In the original paper I tried to compute the all-important ratio beta ANALYTICALLY by an infinite series. But in copying one line to another I unconsciously raised some expression in brackets to a (3/2) power because every other time it had occurred it was raised to that power, when it should not have been raised to any power!

Worse yet, the resultant mistaken first term in the infinite series was numerically what I had expected, so without investigating the rate of convergence of the series I mistakenly assumed (which often does happen in celestial mechanics in the case of "small-parameter perturbations") that I did not need the other terms and quoted that as an estimate of "the answer."

Since then I have solved the relevant integral equations NUMERICALLY, using the highly accurate "Lobatto quadrature" algorithm offered by MATLAB, and my new Abstract, attached, gives the correct answers based on that.

Perhaps the most invaluable thing that any scientist can obtain is intellectually honest & unbiased "peer review." (Too much of it is just "sneer review" devoted to upholding the reigning consensus rather than the search for objective truth.)

I shall be eternally grateful to Gordon Emslie, a former student of noted celestial mechanician Archie Roy of Glasgow U, for having studied my paper with sufficient care to see that I would have needed to take about 200 terms in the infinite series to get adequate accuracy in the answer!
[The terms alternate in sign and the whole sum converges VERY slowly.]

The paper needs to be "fixed" by my going back to the obsolete mathematical-equation software (now out of business) called ChiWriter and retyping two pages, and then converting the resultant mss to Word Perfect and then converting that to MS Word and then using horribly slow Equation Writer to "fix" the inevitable mistakes introduced by conversions, and then resubmitting to a journal.

Mathematical typing is so labor-intensive ("hours per page rather than pages per hour") that I daren't start unless certain that I can spend at least a whole week or two on the task without any interruptions.

Meanwhile I stand by ALL of the basic concepts in the paper, including the two simultaneous integral equations which I had "solved" [for distal multiplier beta and phase-shift phi] by a flawed short-cut, but now solved by numerically minimizing [to zero] the sum of their squares.

I shall insert at the conclusion below the half-dozen MATLAB programs by means of which I computed the correct answers for the new natural constants beta & theta leading to just a handful of possible resonances in the planar Keplerian 3-Body Problem which provide genuine orbital stability.

There is another problem also. I have a much younger friend and collaborator, Antonio Del Popolo (who is widely published in astrophysics) and who was collaborating with me in adding some "updating" literature references to the paper and who submitted it to a journal where the Editors &/or Referee(s) failed to catch the error and it has now been printed without the correction (about which I had regrettably failed to inform him in time)! A copy is attached but must be read in light of the now revised Abstract [BodeBlurbRvsd.doc, attached].

I will paste in below my more detailed letter to Antonio.

I have been invited to a conference in Cambridge U next year and hopefully I can somehow (God willing!) find the time before then to correct the paper and resubmit it to the journal.

-----Original Message-----
From: Dr. Robert W, Bass; Sent: Monday, November 21, 2005 3:54 PM
To: 'Antonino Del Popolo'
Subject: My horrible [yet fixable] mistake re Bode's Law

Antonio,

I do VERY much appreciate your kind efforts but now we have a new problem to deal with. Perhaps you never realized that I had found a horrible [yet fixable] mistake in that paper! I had _thought_ that I had notified you of my need to revise the paper before it was published!! I knew that you had posted it on the LANL arXiv website but I was planning to substitute that with a revised version whenever I could get the time, which unfortunately never happened!

My horrible mistake came about because in copying algebraic formulae from one place to another I got in the habit if raising a certain factor to a 3/2 power. But at a critical place it was NOT raised to that power! As a result, when I expanded the infinite series it looked numerically to me like I got what I was looking for after only two terms and I neglected all of the other terms in an infinite series!

But a friend of mine, Gordon Emslie, then the Dean of the Grad School at UAH [to whom I shall be "eternally grateful!!"] noticed that the infinite series converged so slowly that I was deceiving myself by truncating from the 3rd term onward. When I checked his point I agreed with him that the CORRECTED series converged so slowly that there are needed about 200 terms of the series to get the numerical accuracy needed!

After some study I realized that the only way to get a correct answer was NOT to try to sum a slowly-convergent infinite series but to stop just before I introduced the infinite series and then solve the relevant integral equations (37a,b) numerically, which I did using a supposedly "hyper-accurate" Lobatto Quadrature algorithm in MATLAB.

Unfortunately, this integral is VERY sensitive to tiny numerical parameter-variations! When I plot the Integrand in MathCAD, it looks exactly the same as when I plot it in MATLAB, but the numerical answers are radically different. When I asked MathCAD, they simply ASSUMED that the Lobatto Quadrature with MATLAB would have been "more correct" and asked me to send them my example so that they could improve/correct MathCAD, but I have been too busy to attend to this so far.

Anyway, believing the MATLAB result leads to the following conclusion:

As a result of finding ALL solutions of the relevant integral equations there is not just one Bode factor but FIVE possible such factors (and, assuming the numerical solution is accurate) no more! Four of the factors agree to two significant figures with my self-deceptive result beta = 1.79, but the fifth is radically different!

I would have to rewrite a couple of pages in the middle of the paper but otherwise I stand by everything in the paper!

Unfortunately it would take me about a week to go back and use the obsolete mathematical typesetting package that I had used before [ChiWriter, now out of business], and then save that in Word Perfect, and then use a FREE online facility to convert that to MSWord, and then correct the mistakes introduced by this circuitous route to get a "corrected" paper in .doc form that could be submitted to a contemporary journal.

The CORRECT Abstract is that attached.

As soon as I can find a free week or two we need to draft and send in a CORRECTION to the journal.

Unfortunately, after being involuntarily unemployed for 9 months in 1994 and exhausting my savings, for a relatively short while [10 months in 2005] I was earning nearly $2,000/week [but gave most of the surplus to my children so they could move from CA to NC] but now I am involuntarily unemployed again and trying to "survive" on less than $200/week from teaching at night for F.I.T., and of necessity I am spending all of my time scrambling around trying to find some new source of income. (Also I had asked one of my daughters not to use the Credit Cards I had given her until further advance permission from me, but she just had my son-in-law write me that they were unprepared when the weather suddenly dropped from mid-70s to below freezing every night and they had to use the Credit Cards [which I'm now unable to repay] on an emergency basis to buy winter coats for her and my grandchildren.) So I cannot take the time now to do anything much but look for another source of income!

The paper "looks beautiful" in typeset and archival-journal printed form, but I wish that I had caught up with you before you had succeeded in getting it past an (evidently too lenient!) referee!

AAARRRGGHHHH!

As the Bible says, "pride goeth before a fall!"

Simultaneously thankful/regretful,

Bob

-----Original Message-----
From: Antonino Del Popolo [mailto:antonino.delpopolo@unibg.it]
Sent: Monday, November 21, 2005 12:15 PM
To: Dr. Robert W, Bass

Dear Bob,
I tried again in several journals to have that paper published but with no success.
They published another paper of yours, the one dealing with Bode's law.
I attach it.
Ciao,
Antonio

In the following pages I shall paste in .txt versions of the 6 MATLAB functions used to obtain the CORRET values of beta & theta presented in the revised Abstract, and the numerical values obtained by use of these MATLAB programs to solve the integral equations determining beta & theta.

If you have access to an authorized copy of MATLAB, just copy and paste these programs, one at a time, into the MATLAB Editor as if you had written them yourself, and then run the main program called GraphNewErr.

The amazing thing is that these values of beta & theta are NEW NATURAL CONSTANTS, like pi and Napier’s e, which are INDEPENDENT of the Newtonian constant of gravity G and the masses of the 3 particular point-particle masses involved!

```
function [betOPT,thetOPT,errJK,J,K,thet,bet,err] = GraphNewErr(mk,mm,bet0,betN,thet0,thetN,N);
%********************************************************************
%
%    Two-dimensional Surface providing error criterion to be minimized
%
%   INPUTS
%
%     mk & mm are positive integers specifying a trial resonance
%     such as (2,1),(5,2),(3,1) which by Kepler's Law, for a 2-body
%     problem, would specify distal ratios of 1.5874, 1.8420, 2.0801;
%     but this program refers to a THREE-body problem!
%
%     bet0 < betN specify trial range for distal-ratio Beta
%     thet0 < thetN specify trial range for phase-shift Theta
%     N is an integer specifying number of trial values of Beta & Theta
%        which range _independantly_ over the entire specified rectangle
%
%   OUTPUTS
%
%     err is an N-by-N matrix of (F^2 + G^2) where each definite
%        integral F & G has as inputs (mk,mm) and bet(j), thet(k)
%        for arbitrary (J,K) on 1 </=  J  </= N, 1 </=  K  </= N.
%
%     thet, bet are column N-vectors of the cited trial-values;
%     (J,K) are the values providing the minimum error of errJK;
%     betOPT = bet(J), thetOPT = thet(K) are optimal Beta & Theta
%
%*******************************************************************
format compact
ndx = (1:N)';
ndx = ndx - 1;
thet = thet0 + (thetN - thet0)*ndx/(N-1);
bet = bet0 + (betN - bet0)*ndx/(N-1);
err = zeros(N,N);
for j = 1:N
  for k = 1:N
    kount = j + k
    F1 = F(mk,mm,bet(j),thet(k));
    G1 = G(mk,mm,bet(j),thet(k));
    [errjk] = F1^2 + G1^2;
    err(j,k) = errjk;
  end
end
[errminJK,JK] = min(err);       % row with min err of each col
[errminabs,K] = min(errminJK)   % K is col with min err
[errminabs2,J] = min(err(:,K))  % J is row with min err
errJK = err(J,K);
check = abs(errJK - errminabs)+abs(errminabs - errminabs2)  % = 0 ?
betOPT = bet(J)
thetOPT = thet(K)
% end of GraphNewErr.txt
```

```
function [psi1] = PSI(sig,bet,thet,mk,mm);
%****************************************
%
%  factor of integrand in both F & G
%
%****************************************
format compact
epsq = 2*bet/(1 + bet*bet);
xi = (mk - mm)*sig + thet;
den = (1 - epsq*cos(xi)).^(3/2) + eps;
psi1 = 1./den;
% end of PSI.txt

function [F1] = F(mk,mm,bet,thet);
%*************************************************
%
%   first integral [using Lobatto quadrature, quadl]
%
%*************************************************
format compact
fctr = 1/(2*pi);
F1 = fctr*quadl(@fnF,0,2*pi,[],[],mk,mm,bet,thet);
% end of F.txt

function [fnF1] = fnF(sig,bet,thet,mk,mm);
%************************************
%
%       integrand of F
%
%************************************
format compact
psi1 = PSI(sig,bet,thet,mk,mm);
fctr = bet*cos(mm*sig) - cos(mk*sig + thet);
fnF1 = sin(mk*sig).*fctr.*psi1;
% end of fnF.txt

function [G1] = G(mk,mm,bet,thet);
%**************************************************
%
%   second integral [using Lobatto quadrature, quadl]
%
%**************************************************
format compact
fctr = 1/(2*pi);
G1 = fctr*quadl(@fnG,0,2*pi,[],[],mk,mm,bet,thet);
% end of G.txt
```

```
function [fnG1] = fnG(sig,bet,thet,mk,mm);
%************************************
%
%      integrand of G
%
%************************************
format compact
psi1 = PSI(sig,bet,thet,mk,mm);
fctr = bet*sin(mm*sig) - sin(mk*sig + thet);
fnG1 = sin(mk*sig).*fctr.*psi1;
% end of fnG.txt
```

```
==========================================================================
--------------------------------------------------------------------------
Resonance    KeplerBeta   |    BassBeta          BassTheta       MaxResidual
 (mk,mm)    (mk/mm)^[2/3] |
                             |
(2,1)        1.5874      |  1.70716761693332   0.86689698316336    2/10^15
                             |
(3,1)        2.0801      |  1.77317834465894   0.90780670929085    3/10^16
                             |
(4,1)        2.5198      |  1.80915773588908   0.92444093688167    1/10^15
                             |
(3,2)        1.3104      |  2.70842603143354   1.47431503470921    2/10^15
                             |
(5,2)        1.8420      |  1.77256086302991   0.68613875674636    3/10^12
--------------------------------------------------------------------------
=========================================================================
```

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DYNAMICAL DERIVATION OF BODE'S LAW

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In a planetary or satellite system, idealized as n small bodies in an initially coplanar with concentric orbits around a large central body obeying the Newtonian point-particle mechanics, resonant perturbations will cause a dynamical evolution of the orbital radii except for cases with highly specific mutual relationships. In particular, the most stable situation can be achieved only when each planetary orbit is roughly twice as far from the Sun as the preceding one. This has been empirically observed by Titius (1766) and Bode (1778). By reformulating the problem as a hierarchical sequence of (unrestricted) 3-body problems and considering only the gravitational interactions among the central body and the body of interest and the adjacent outer body in the orbits, it is proved that the resonant perturbations from the outer body will destabilize the inner body (and vice versa) unless its mean orbital radius is a unique and specific multiple of β, the distal multiplier, of the inner body. In this way a sequence of concentric orbits can each stabilize the adjacent inner orbit, and since the outermost orbit is already tied to the collection of the inner orbits by conservation of total angular momentum, the entire configuration is stabilized.

Keywords: Planets and satellites; general-solar system; formation.

1. Introduction

The distribution of planets in the Solar system is nonrandom distribution and their mean distance from the Sun, when numbered from the center, form a rough geometric progression:

$$\frac{r_{n+1}}{r_n} \approx 1.75 \tag{1}$$

or

$$r_n = 0.4 + 0.15 \times 2^n\,, \quad n = -\infty, 1, \ldots, 8\,. \tag{2}$$

The expression (2) is known as the Titius–Bode law, and it roughly describes the planetary semi-major axes in astronomical units with Mercury assigned $i = -\infty$, Venus $i = 1$, Earth $i = 2$, etc. Usually the asteroid belt is assigned $i = 4$. The law

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fits the planets Venus, Earth, Mars, Jupiter, Saturn and Uranus quite well, and successfully predicts the existence and the locations of Uranus and the asteroids. However, the law breaks down for the case of Neptune and Pluto. There is no reason why Mercury should have $i = -\infty$ rather than $i = 0$ except that it fits better in this way. In addition, the total mass of the asteroid belt is much smaller than the mass of any planet, so it is not clear why it should be counted as 1. When this relationship was discovered by Titius in 1766 and was published by Bode six years later, it showed good agreement with the actual mean distances of the planets known at that time, namely: Mercury (0.39 AU), Venus (0.72 AU), Earth (1.0 AU), Mars (1.52 AU), Jupiter (5.2 AU), and Saturn (9.55 AU). Uranus, discovered in 1781, has mean orbital distance 19.2 AU, which also agrees with the prediction. The asteroid Ceres, discovered in 1801, has mean orbital distance 2.77 AU, which fills the apparent gap between Mars and Jupiter. However, Neptune, discovered in 1846, has mean orbital distance 30.1 AU, and Pluto, discovered in 1930, has mean orbital distance 39.5 AU; these are large discrepancies from the predicted value by Bode's law, 38.8 AU and 77.2 AU, respectively. It is thought that the outer planets do not fit as well due to millions of tears of comet impacts and a looser gravitational hold. Also, Pluto is thought to have been created outside our solar system and the law would therefore not apply to it.

In the past 200 years, there was a debate in astronomy as whether the accuracy of Bode's Law is simply chance or the law underlines an unseen force in the creation of the solar system. A fairly comprehensive history up to the year 1971 of the law and the attempts in explaining it can be found in Nieto's book.[43] Some theories of the origin of the solar system have tried to explain the apparent regularity in the mean orbital distances of the planets, arguing that it could not arise by chance but must be a manifestation of the laws of physics. There are astronomers who conclude, based on the predicted positions, that the planets Neptune and Pluto are no longer at their original positions. However, as the Bode's law is not a law in the usual scientific sense, i.e., it is not universal and invariant, it should not be taken as evidence for such a conclusion. On the assumption that planets need a certain amount of space to form without competing for material, computer simulations with random numbers have shown that, for early solar systems, the planet spacing follows the similar laws.[44] Despite this, many scientists have come up with explanations for the law over the centuries. A theory presented by Stuart Weidenchilling and Donald Davis claims that the planets ended up where they are due to the frictional drag that brings the smaller planets closer to the Sun and due to the "gravitational perturbations from embryonic planets." Under the right conditions there would be favorable locations for planets to form.[45] Another recent theory employs a simple numerical sequence that combines scale and rotational invariances — the idea of which a physical parameter such as density when reaches a maximum or a minimum will always follow a simple relationship. Using this insight, Francois Graner and Berengere Dubrulle have improved the Bode's Law to give better predictions. They claim that to believe in Bode's Law is to believe in scale invariance in the early solar system.[46] Summarizing the most modern arguments concerning the validity of Bode's law we can classify them as one of the followings:

(i) Attempts to elucidate the physical processes leading to the Bode' law; these are based on a variety of mechanisms, including dynamical instabilities in the protoplanetary disk,[47–49] gravitational interactions between planetesimals,[44] or long-term instabilities of the planetary orbits.[50–52]
(ii) Discussions that ignore the physics but try to assess whether the success of the Bode's law is statistically significant;[53–55] conclusions that go from the Bode's law being real rather than artifactual to the contrary. As observed by Hayes and Tremaine,[56] these conclusions are flawed because of the assumptions made in the analyses.
(iii) Discussions of other laws that may influence the spacing of the planets. Many of these involve resonances between the mean motions of the planets, such as Ref. 57 (see also Refs. 58–60).

For what concerns in item (iii), the commensurability and resonance phenomena of the solar system motion structure — including planets, asteroids, planet's satellites — have been the object of detailed discussions and experimental examination in the past years. Systematic observations and measurements have been carried out using all available means. A considerable bulk empirical data for the solar system has been obtained from the "Voyager 2" mission. The understanding of the inevitable resonance behavior of evolving mature oscillation systems leads to series of interesting concepts about the resonance behavior of the motion dynamics of the Solar system. One of the most outstanding example among them is Molchanov's hypothesis about the complete resonance behavior of the large planets in-orbit motion. A. M. Molchanov noticed that the mean motion of the nine large planets are related approximately to nine linear homogeneous equations

$$k_1^{(j)} n_1 + \cdots + k_9^{(j)} n_9 = 0 \,, \quad j = 1, \ldots, 9 \tag{3}$$

with the integer coefficients

$$k_1^{(j)}, \ldots, k_9^{(j)} \,. \tag{4}$$

The mean motions of the Jovian, Saturn and Uranian satellites are related to each other by the similar equations. If asteroids are considered as planets (ecept for Pluto), their planetary distances obey the following regularity:[61]

$$\frac{a_{k+1}}{a_k} \cong 1.75 \pm 0.20 \,, \tag{5}$$

i.e. the relation of the semi-major orbital axes of the neighboring planets is almost constant.

In this paper, we perform a dynamical derivation of the Bode's law. The basic strategy is as follows: the famous "Poincaré map" which is often used by physicists with numerical integration,[34] will be implemented analytically. In the present context, this means that an arbitrary line-segment will be defined transversely to a periodic generating orbit, and the nearby orbits will be followed until this one-dimensional "surface of section" is intersected a second time. This defines a continuous mapping of the section into itself. The initial periodic solution of this map is now a fixed point. The behavior of nearby orbits can now be studied by considering the simpler problem of iterating this map in a small neighborhood of the fixed point. It will be proved below that no fixed point exists (i.e. no periodic solution exists) unless at $\mu = 0$ the distal ratio defined in the abstract is precisely β_0.

2. Problem Formulation

Let $x^i \in \mathbb{E}^2$ $(i = 1, 2, 3, \ldots, n)$ denote 2-vectors or elements of two-dimensional real Euclidean space $\mathbb{E}^2$ which represent the positions of n point-particles of masses M_i in orbit around a central body of mass M_0 and at position x^0. Let $v^i \in \mathbb{E}^2$ denote their velocities $\dot{x}^i \equiv dx^i/dt$ with respect to time t, where $\dot{} := d/dt$. Let $x \cdot y \equiv (x, y)$ denote the *scalar product* between any two vectors x, y, and let $\|x\| := (x, x)^{1/2}$ denote the *norm* (length) of any vector.

It is assumed that $\mu_i := M_i/M_0 \ll 1$. Now rescale the masses so that $M_0 = 1$ and the smaller masses $\mu_i \ll 1$. Let $\mu_{\max}$ denote the largest of small masses, and rewrite them as

$$\mu_i = \mu \cdot \mu_i^0 = \mu \cdot \varepsilon_i^0 \cdot \mu_{\max}\,, \quad 0 \le \mu \le 1\,, \quad (0 < \varepsilon_i^0 \le 1)\,, \tag{6}$$

so that μ is the *perturbation parameter.* All that will be *proved* here is for μ 'sufficiently small,' though there are good reasons for believing that an analytical continuation (in the manner of Poincaré) can be made all the way from $\mu = 0$ to $\mu = 1$; this is because Leray–Schauder index of the 'generating solution' at $\mu = 0$ will be *proved* to be unity, and the fact that this *integer-valued* topological invariant, specifying in some sense the true multiplicity of actually existing solutions, is a *continuous* homotopy invariant and so can change its value *discontinuously*, i.e. terminate, *only* at a bifurcation or singularity of the solution (i.e. a collision or an ejection), demonstrates that the homotopy on $\mu \in [0, 1]$ is legitimate *unless* there is an intermediate value of $\mu < 1$ at which there is a collision between two bodies or one body is ejected to infinity; this makes rigorous Strömgren's *principle of natural termination*, discovered empirically by numerical integration,[1] in which a family of periodic solutions being studied by the variation of a parameter can cease to exist *only* at an ejection/collision event.

Now introduce *relative* coordinates, in which each x^i is replaced by $(x^i - x^0)$, although for convenience the latter will be renamed as x^i; it is well known that the system becomes, for the indices $i = 1, 2, \ldots, n$,

$$\dot{x}^i = v^i\,, \tag{7a}$$

$$\dot{v}^i = -G \cdot (1 + \mu_i) \cdot \left(\frac{x^i}{\|x^i\|^3}\right) - G \cdot \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_i \cdot \left(\left[\frac{x^j}{\|x^j\|^3}\right] - \left[\frac{(x^j - x^i)}{\|(x^j - x^i)\|^3}\right]\right), \tag{7b}$$

where the absolute acceleration of the origin of the coordinate system which now coincides with the position of the Sun or the central body, is the source of the first term in the summation (cf. Ref. 24, pp. 84 and 85), and where G denotes the Newton–Cavendish parameter or the gravitational constant.

Now, following Ref. 24, assume that each planet is affected significantly only by the Sun and the next further outward planet:

$$\begin{gathered}\dot{v}^k = -G\cdot(1+\mu_k)\cdot\left(\frac{x^k}{\|x^k\|^3}\right) + G\cdot\mu_k\cdot\left(\left[\frac{(x^m-x^k)}{\|x^m-x^k\|^3}\right] - \left[\frac{x^m}{\|x^m\|^3}\right]\right),\\ k=(i,j)\,,\quad j=(i+1)\,,\quad m=m(k)\,,\\ m(i)=j\,,\quad m(j)=i\,,\quad (i=1,2,\ldots,n-1)\,.\end{gathered} \tag{7c}$$

Again, following Ref. 24, we shall solve the Newtonian system (7a), (7c) by successive approximations but in a modified manner. Note that if $\mu_k = 0$, then (7c) can be solved using the concentric circular solutions, with frequencies ω_k given by Kepler's third law as

$$(\omega_k)^2 = G\cdot(1+\mu_k)/(\rho_k)^3\,,\quad \rho_k := \|x^k\|\,. \tag{8}$$

For the purpose of successive approximations, replace (7c) by the equivalent *ordinary differential equation* (ODE)

$$\begin{aligned}\dot{v}^k &= -(\omega_k)^2x^k + G\cdot\mu_k\cdot\left(\left[\frac{(x^m-x^k)}{\|x^m-x^k\|^3}\right] - \left[\frac{x^m}{\|x^m\|^3}\right]\right)\\ &\quad + [(\omega_k)^2 - G\cdot(1+\mu_k)/\|x^k\|^3]\cdot x^k\,,\end{aligned} \tag{9}$$

which clearly show that the system is just a perturbation of the *harmonic oscillator* problem.

We shall need the following lemma, which is apparently the modification of the well-known basic results of the ODE theory.[25–28]

3. Lemmas from ODE Theory

Definition. An m-vector function $f : \mathbb{E}^m \to \mathbb{E}^m$ has a global Lipschitz constant κ if there is a positive real number $\kappa > 0$ such that, for all $z^i \in \mathbb{E}^m$,

$$\|f(z^2) - f(z^1)\| \le \kappa\cdot\|z^2 - z^1\|\,. \tag{10}$$

Theorem. *Consider the m-vector ODE initial condition problem (ICP)*

$$\dot{z} = f(z)\,,\quad z(0) = z^0\,,\quad (0 \le t \le T)\,, \tag{11}$$

where f satisfies a global Lipschitz condition, and where the maximum time T is arbitrary but finite. Let A denote an arbitrary $m\times m$ constant real matrix, and define the vector function g by $g(z) := f(z) - Az$, which also is globally Lipschitzian, with Lipschitz constant now $[\kappa + \|A\|]$, where $\|A\|$ denotes the Euclidean norm of A.

Now construct the sequence of functions $\{z^k(t)\}$ *by successive solution of the forced linear linear ODE*

$$\dot{z}^k = Az^k + g(z^{k-1}(t))\,, \quad z(0) = z^0\,, \quad (k = 2, 3, \ldots)\,, \tag{12}$$

where the initial iterate $z^1(t)$ *is any arbitrary continuous function of* t *on* $[0, T]$ *such that its initial value* $z^1(0) = z^0$; *or, equivalently, construct the sequence by*

$$z^k(t) = \exp(At) \cdot z^0 + \int_0^t \exp(A[t - \tau]) \cdot g(z^{k-1}(\tau))d\tau\,, \quad 0 \le t \le T\,, \tag{13}$$

i.e. by integrating the right-hand side to obtain the left-hand side, and then inserting the result into the right-hand side and repeating the operation. The resulting sequence is guaranteed to converge uniformly on $[0, T]$ *to the exact and unique solution of* (11), *no matter what is the arbitrary initial iterate* $z^1(t)$, *and no matter what is the arbitrary matrix* A, *and no matter how large is the* (*fixed*) *time interval* $[0, T]$.

Proof. Any standard work on ODE's will show how to use Lagrange's method of *variation of constants* to prove that the ODE ICP

$$\dot{z} = Az + g(z)\,, \quad z(0) = z^0\,, \quad (0 \le t \le T)\,, \tag{14}$$

is completely equivalent to the Volterra integral equation problem

$$z(t) = \exp(At) \cdot z^0 + \int_0^t \exp(A[t - \tau]) \cdot g(z(\tau))d\tau\,, \quad 0 \le t \le T\,, \tag{15}$$

where

$$\exp(At) := \sum_{k=0}^{\infty} A^k \cdot \frac{t^k}{k!}\,. \tag{16}$$

Therefore, if the sequence in (12) and (13) converges uniformly, then the result satisfies (15) and so is the solution of (11). Because every continuous function on a closed and bounded subset of a finite-dimensional Euclidean space has (and assumes) a finite maximum and minimum, there exists (on the closed, bounded subset $[0, T] \subset \mathbb{E}^1$)

$$\phi_0 = \max_{t \in [0,T]} \{\|z^1(t) - z^0(t)\|\}\,. \tag{17}$$

By repeatedly applying the triangle inequality to (16) one finds readily that

$$\|\exp(At)\| \le \sum_{k=0}^{\infty} \|A\|^k \cdot \frac{t^k}{k!} \equiv \exp(\|A\| \cdot t) \le \gamma := \exp(\|A\| \cdot T)\,. \tag{18}$$

Accordingly it is easy to prove by induction that (setting $\kappa_1 = \kappa + \|A\|$)

$$\|z^{k+1}(t) - z^k(t)\| \le \phi_0 \cdot \frac{(\gamma \cdot \kappa_1 \cdot t)^k}{k!}\,, \tag{19}$$

$$\lim_{N \to +\infty} \{\|z^N(t) - z^0(t)\|\} \le \sum_{k=0}^{\infty} \|z^{k+1}(t) - z^k(t)\| \le \phi_0 \cdot \exp(\gamma \cdot \kappa_1 \cdot T)\,, \tag{20}$$

for each $t \in [0, T]$, so that $\{z^k(t)\}$ is a Cauchy sequence (for each t) and its limit $z(t)$ must exist. Consideration of the approximating sums $\sum_N$ and $\sum_{N+1}$ to the series in (20) shows that the convergence is uniform. $\square$

In the sequel, we shall frequently use the 2×2 identity matrix I_2 and its 'imaginary' skew-symmetric counterpart $J_2 \equiv -(J_2)'$ defined as

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{21a}$$

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{21b}$$

$$(J_2)^2 = -I_2 \,. \tag{21c}$$

For present purposes, the result that

$$\exp(J_2 \cdot \phi) \equiv \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{22}$$

will be all-important. (To prove it, insert (21c) into (16).) A similar result which will be needed is that if the 4×4 matrix A is defined by (using MATLAB notation) $A = [0, I_2; -\omega^2 I_2, 0]$, then it is easy to prove by induction that

$$A^{2k} = (-1)^k \cdot \omega^{2k} \cdot I_4 \,, \qquad A^{2k+1} = (-1)^k \cdot \omega^{2k} \cdot A \,, \tag{23}$$

whence from (16) it is immediate that (letting I denote I_2)

$$\begin{aligned} \exp(A \cdot \phi) &= \exp \begin{pmatrix} 0 & I \cdot \phi \\ -\omega^2 \cdot I \cdot \phi & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega \cdot \phi) \cdot I & [1/\omega] \cdot \sin(\omega \cdot \phi) \cdot I \\ -\omega \cdot \sin(\omega \cdot \phi) \cdot I & \cos(\omega \cdot \phi) \cdot I \end{pmatrix} \end{aligned} \tag{24}$$

4. Principal Result

Theorem. *Consider the semi-restricted Copernican–Newtonian* $(n+1)$*-body problem as formulated in* (7a) *and* (9), *with initial conditions* (*ICs*) *corresponding to Keplerian concentric circular orbits*, *i.e. in which*

$$x^k(0) \cdot v^k(0) = 0 \,, \tag{25a}$$

so that 3 *parameters suffice to specify the ICs. Let these parameters be defined by the triplet* (ρ, θ, ω) *given by polar coordinates in which* ρ *denotes the initial orbital radius*, θ *the initial phase angle*, *and* ω *the initial angular velocity*, *assumed to conform to Kepler's law* (8), *which reduces the IC to a* 2*-parameter set* (ρ, θ). *Specifically*, *assume* (8) *and let the ICs of* (7a) *and* (9) *be*

$$\begin{aligned} x^k(0) &= \rho_k(\cos(\theta_k^0), \sin(\theta_k^0))' \,, \\ v^k(0) &= \omega_k \cdot J_2 \cdot x^k(0) \equiv -\omega_k \cdot \rho_k(\cos(\theta_k^0), \sin(\theta_k^0))' \,, \end{aligned} \tag{25b}$$

where $'$ denotes vector-matrix transposition. Suppose further that the ICs are so chosen that the frequencies are resonant as in (1); *for later convenience, we may assume that*

$$1 \leq m_i\,, \qquad m_j \leq 5\,, \tag{26}$$

although the main result holds when these integers are arbitrarily large, which means that the initial frequencies (ω_i, ω_j) *are essentially arbitrary. Now, keeping the* resonant *frequencies* (ω_i, ω_j) *found at* $\mu = 0$ fixed, *consider the variation of* (ρ, θ) *as a function of* μ *in such a way as to preserve the periodicity* (*namely,* "*isoperiodic*" *continuation*) :

$$x^i(t+T) \equiv x^i(t)\,, \quad T = 2 \cdot \frac{\tau}{\omega}\,, \quad \omega = \frac{\omega_i}{m_i} = \frac{\omega_j}{m_j}\,, \tag{27}$$

while ignoring the question of periodicity of x^j, $j = i+1$. *A necessary and sufficient condition that for sufficiently small* μ *there exist* $(\rho_i(\mu), \theta_i(\mu))$ *preserving the periodicity* (27) *is that there exist a phase-shift* $\phi = \phi(\mu)$ *and a distal multiplier* $\beta = \beta(\mu)$ *such that*

$$\theta_{i+1}(\mu) = \theta_i(\mu) + \phi\,, \tag{28}$$

$$\rho_{i+1}(\mu) = \beta \cdot \rho_i(\mu)\,, \tag{29}$$

where, in the second post-Keplerian approximation (ϕ, β) *are given by*

$$\phi = m \cdot \left(\frac{\pi}{2}\right) + \cdots\,, \quad (m = 1, 2, 3, \ldots)\,, \tag{30a}$$

$$\beta = \beta_0 + \cdots \equiv \frac{1}{\sqrt{(3/2)^{2/3} - 1}} + \cdots = 1.794980 + \cdots\,, \tag{30b}$$

whence, by Kepler's law in the form

$$r \equiv \frac{m_i}{m_{i+1}} = \beta^{3/2} + \cdots = 2.40 + \cdots\,, \tag{30c}$$

the only possibilities for the resonance in the low-order case (26) *are*

$$m_i : m_{i+1} = 2 : 1 \text{ or } 5 : 2 \text{ or } 3 : 1\,. \tag{30d}$$

Remark. Canceling the arbitrary *supplementary* assumption (26) leads to (30d); the initial generating orbits can then be placed arbitrarily closely to *any* pair of concentric circles with arbitrary radii (because of an arbitrary irrational β in (29), and so its (3/2)th power r as in (30c) can be approximated as closely as desired by a rational number m_i/m_{i+1}). In this way it can be seen that the *unique* distal multiplier β given in (30b) as 1.80 at $\mu = 0$ is a *constant* which is completely independent of the assumed trial value of β at $\mu = 0$!

Proof. Let e^i denote the columns of $I_2 = (e^1, e^2)$. Then at $\mu = 0$ the solution of the given ODE ICP is

$$x^k(t) = \rho_k \cdot \exp(J_2 \cdot \theta_k) \cdot e^1\,, \quad \theta_k = \omega_k \cdot t + \theta_k^0 = m_k \cdot \sigma + \theta_k^0\,, \quad \sigma := \omega \cdot t\,, \tag{31}$$

We want this to be the linear part of the reformulation presented in the lemma, so choose A as in (24), and then reexpress the problem in the equivalent form (15), where the required matrix exponential is given by (24), here now $z = (x', v')' \in \mathbb{E}^4$ and, corresponding, the $g(z)$ in (15) is given by $g = (0', [g^{km}]')'$, where

$$g^{km} = G \cdot \mu_k \cdot \left(\left[\frac{x^m - x^k}{\|x^m - x^k\|^3} \right] - \left[\frac{x^m}{\|x^m\|^3} \right] \right) \\ + [(\omega_k)^2 - G \cdot (1 + \mu_k)/\|x^k\|^3] \cdot x^k \,. \tag{32}$$

Thus, define the 'osculating' *generating solution* at $\mu = 0$ by

$$x^{k,0}(t) = \cos(\omega_k \cdot t) \cdot x^k(0) + \left[\frac{1}{\omega_k} \right] \cdot \sin(\omega_k \cdot t) \cdot v^k(0) \,, \tag{33a}$$

$$v^{k,0}(t) = \cos(\omega_k \cdot t) \cdot v^k(0) - \omega_k \cdot \sin(\omega_k \cdot t) \cdot x^k(0) \,. \tag{33b}$$

Now the problem is rigorously equivalent to solving the integral equation

$$x^k(t) = x^{k,0}(t) + \int_0^t \left[\frac{1}{\omega_k} \right] \cdot \sin(\omega_k[t - \tau]) \cdot g^{km}(x^k(\tau), x^m(\tau)) d\tau \,; \tag{34a}$$

although the formulation (24) gives a second equation

$$v^k(t) = v^{k,0}(t) + \int_0^t \cos(\omega_k[t - \tau]) \cdot g^{km}(x^k(\tau), x^m(\tau)) d\tau \,. \tag{34b}$$

The second equation is a consequence of differentiating the first equation (34a) with respect to time t, and therefore is of no further consequence, after noting that in (33a) the initial velocity $v^k(0) = \omega_k \cdot J_2 \cdot x^k(0)$ is already defined in terms of $x^k(0)$ by hypothesis. □

It is a well-known consequence of basic ODE theory that the (necessarily unique) solutions of (7a) and (9) for $\mu = 0$, namely (33), and the solutions for sufficiently small $\mu > 0$ remain arbitrarily close together for any finite time, specifically here for $0 \leq t \leq T = 2 \cdot \pi/\omega$, provided only that μ be sufficiently small. Therefore for sufficiently small μ one knows *a priori* that the solutions remain within planar concentric annuli surrounding the circular orbits (33), and therefore do not approach each other during the time of interest. Accordingly, one may find a global Lipschitz constant for such relevant domains of the Cartesian products of $\mathbb{E}^4$ (in which the Jacobian matrix of the right-hand side of (11) is continuous, and so has a bounded norm κ in the relevant z-domain). Therefore the lemma is applicable, for sufficiently small μ, and so the solution of (34) exists and can be constructed by successive approximations (with $j = 1, 2, 3, \ldots$):

$$x^{k,j}(t) = x^{k,0}(t) + \int_0^t \left[\frac{1}{\omega_k} \right] \cdot \sin(\omega_k[t - \tau]) \cdot g^{km}(x^{k,j-1}(\tau), x^{m,j-1}(\tau)) d\tau \,, \tag{35}$$

taking as the initial iterate the decoupled Keplerian solution (33), for $0 \leq t \leq T$.

Next, look at the solution of (34) only at every revolution of duration $T = 2\cdot\pi/\omega$, wherein the *commensurability* of ω_i and ω_{i+1} ensures that there is a *common period* to the initial iterates of the two adjacent orbits. This gives the famous *Poincaré map*:

$$x^k(T) \;=\; x^k(0) - \mu\cdot\left(\frac{G\mu_m^0}{[m_k\omega^2]}\right)\cdot\int_0^{2\pi}\sin(m_k\sigma)\cdot h^{km}(x^k(\sigma),x^m(\sigma))d\sigma\,, \tag{36a}$$

$$h^{km} := \left(\left[\frac{(x^m - x^k)}{\|x^m - x^k\|^3}\right] - \left[\frac{x^m}{\|x^m\|^3}\right]\right) + [(\omega_k)^2 - Gp\cdot(1+\mu_k)/\|x^k\|^3]\cdot x^k\,. \tag{36b}$$

We shall prove that this map can be expressed in the form

$$x^k(T) = x^k(0) - \mu\cdot\left(\frac{G\mu_m^0}{[m_k\omega^2]}\right)\cdot f^{km}(\rho_k,\theta_k^0,\rho_m,\theta_m^0;\mu)\,, \tag{37a}$$

$$f^{km}(\rho_k,\theta_k^0,\rho_m,\theta_m^0;\mu) := \int_0^{2\pi}\sin(m_k\sigma)\cdot h^{km}(x^k(\sigma),x^m(\sigma))d\sigma\,, \tag{37b}$$

where for certain specific values of (ρ_k,θ_k^0) the Jacobian determinant of f^{km} with respect to (ρ_k,θ_k^0) is positive at $\mu = 0$, and where $f^{km} = 0$ for these same specific values, at $\mu = 0$; then, by the implicit function theorem there exist functions $(\rho_k(\mu),\theta_k(\mu))$ for sufficiently small values of μ which satisfy $f^{km} \equiv 0$ and so which provide the desired periodicity of $x^k(T)$.

It will become evident that the conditions to be derived are also necessary, because if they are are not satisfied, f^{km} provides a *resonant* forcing term which drives $x^k(T)$ ever farther from $x^k(0)$, and will do so without bound (or until the resonant 'pumping' drives the map's next iterate out of the map's domain of definition).

To anticipate the results of a rather lengthy and arduous calculation, it will be proved below that f^{km} has the form

$$f^{km} = \eta_{km,m}\cdot\exp(J_2\Phi_{km,m})\cdot x^m(0) - \eta_{km,k}\cdot\exp(J_2\Phi_{km,k})\cdot x^k(0)\,, \tag{38}$$

where the scalar functions $\eta_{kmj}(\mu)$ and $\Phi_{kmj}(\mu)$ have the properties that

$$\eta_{km,m}(0) = 0\,,\quad \eta_{km,k}(0) = \frac{1}{2}\,,\quad \Phi_{km,k}(0) = \frac{\pi}{2}\,, \tag{39}$$

so that at $\mu = 0$ the Jacobian matrix H of f^{km} with respect to $x^k(0)$ is given by

$$H = -\left(\frac{1}{2}\right)\cdot\exp\left(J_2\cdot\left[\frac{\pi}{2}\right]\right) = -\left(\frac{1}{2}\right)\cdot J_2\,,\quad \det(H) = \frac{1}{4} > 0\,, \tag{40}$$

and, as claimed, the Lerray–Schauder index[29] of the chosen generating solution is unity.

It is evident by inspection that upon the second iteration the third term (36b) drops out by the choice of a Keplerian generating solution, so that for consideration of the second iterate of the successive approximations we may simplify (36b) to

$$h^{km} := \left[\frac{(m^m - x^k)}{\|x^m - x^k\|^3} \right] - \left[\frac{x^m}{\|x^m\|^3} \right] . \tag{41}$$

It is also immediate that the second term in (41) may be omitted, because by (31)

$$\begin{gathered} \frac{x^m}{\|x^m\|^3} = \left(\frac{1}{\rho_m^2} \right) \cdot \exp(J_2 \cdot m_m \sigma) \cdot \exp(J_2 \cdot \theta_m^0) \cdot e^1 , \\ \int_0^{2\pi} \sin(m_k \sigma) \cdot \exp(J_2 \cdot m_m \sigma) d\sigma \equiv 0 \end{gathered} \tag{42}$$

because of the hypothesis that $m_k \neq m_m$. Thus we have simplified the calculation to that of evaluation of

$$\begin{aligned} f^{km} &= \int_0^{2\pi} \sin(m_k \sigma) \cdot \Psi(\sigma) \cdot (x^m(\sigma) - x^k(\sigma)) d\sigma , \\ x^k(\sigma) &= \rho_k \cdot \exp(J_2 \cdot [m_k \sigma + \theta_k^0]) \cdot e^1 , \end{aligned} \tag{43}$$

$$\Psi(\sigma) := \frac{1}{\|x^m(\sigma) - x^k(\sigma)\|^3} . \tag{44}$$

Now by the chief property of the scalar product (x, y), namely that if M is an arbitrary matrix $(Mx, y) \equiv (x, M'y)$, and the fact that $J_2' = -J_2$, it is easy to calculate that

$$\|x^m(\sigma) - x^k(\sigma)\|^2 \equiv \|x^m\|^2 + \|x^k\|^2 - 2 \cdot \rho_m \cdot \rho_k (e^1, \exp(J_2 \xi) e^1) , \tag{45a}$$

$$\xi = (m_k - m_m) \cdot \sigma + \theta_k^0 - \theta_m^0 , \tag{45b}$$

$$\Psi(\sigma) = \frac{\psi(\sigma)}{[(\rho_k)^2 + (\rho_m)^2]^{3/2}} , \tag{45c}$$

$$\psi(\sigma) = \frac{1}{[1 - (\varepsilon_{km})^2 \cos(\xi)]^{3/2}} , \quad (\varepsilon_{km})^2 := \frac{2 \cdot \rho_k \cdot \rho_m}{[(\rho_k)^2 + (\rho_m)^2]} , \tag{45d}$$

$$\psi(\sigma) = 1 + \left(\frac{3}{2} \right) \cdot (\varepsilon_{km})^2 \cos(\xi) + \left(\frac{15}{8} \right) \cdot (\varepsilon_{km})^4 \cos^2(\xi) + \cdots , \tag{45e}$$

where the binomial series in (45e) always converges because $(\varepsilon_{km})^2 < 1$ whenever ρ_k and ρ_m are distinct (which is a consequence of the fact that then $0 < (\rho_k - \rho_m)^2$ and of obvious manipulations of the expansion of the latter).

Now, remembering that ψ depends upon both k and m, we define

$$A_{kmj} := \left(\frac{1}{[2\pi]} \right) \cdot \int_0^{2\pi} \sin(m_k \sigma) \cdot \cos(m_j \sigma) \cdot \psi(\sigma) d\sigma , \tag{46a}$$

$$B_{kmj} := \left(\frac{1}{[2\pi]} \right) \cdot \int_0^{2\pi} \sin(m_k \sigma) \cdot \sin(m_j \sigma) \cdot \psi(\sigma) d\sigma , \tag{46b}$$

$$\eta_{kmj} := \left\{(A_{km})^2 + (B_{km})^2\right\}^{1/2}, \tag{46c}$$

$$\Phi_{kmj} := -\arctan\left\{\frac{B_{kmj}}{A_{kmj}}\right\}, \tag{46d}$$

and note that (43) and (44) may be simplified by the use of the novel identity

$$\left(\frac{1}{[2\pi]}\right) \cdot \int_0^{2\pi} \sin(m_k\sigma)\cdot\psi(\sigma)\cdot\exp(J_2[m_j\sigma + \theta_j^0])d\sigma \equiv \eta_{kmj}\exp(J_2[\Phi_{mkj} + \theta_j^0])\,. \tag{47}$$

It is the radical simplification provided by the apparent hitherto unnoticed identity (47) which appears to be the chief innovation in the present work. We may now write that

$$f^{km} = \frac{d^{km}}{[(\rho_k)^2 + (\rho_m)^2]^{3/2}}, \tag{48a}$$

$$\begin{aligned} d^{km} &:= \big(\rho_m\eta_{kmm}\cdot\exp(J_2[\Phi_{kmm} + \theta_m^0]) \\ &\quad - \rho_k\cdot\eta_{kmk}\cdot\exp(J_2[\Phi_{kmk} + \theta_k^0])\cdot e^1 \\ &\equiv \rho_m\cdot\eta_{kmm}\cdot\begin{pmatrix}\cos(\Phi_{kmm} + \theta_m^0)\\ -\sin(\Phi_{kmm} + \theta_m^0)\end{pmatrix} \\ &\quad - \rho_k\cdot\eta_{kmk}\cdot\begin{pmatrix}\cos(\Phi_{kmk} + \theta_k^0)\\ -\sin(\Phi_{kmk} + \theta_k^0)\end{pmatrix}. \end{aligned} \tag{48b}$$

From a mere inspection of (48) it is now evident that the *necessary and sufficient conditions* for f^{km} to vanish are the *orbital resonance* defined by

$$\rho_m = \beta\cdot\rho_k\,, \qquad \beta = \frac{\eta_{kmk}}{\eta_{kmm}}, \tag{49a}$$

and the corresponding enabling phase-shift

$$\theta_m^0 = \theta_k^0 + \phi_{mk} + M\cdot\pi\,, \qquad \phi_{mk} = \Phi_{kmk} - \Phi_{kmm}\,, \quad (M = 1, 2, 3, \ldots)\,. \tag{49b}$$

This completes the easy part of the present derivation.

Now, we begins the hard work of evaluation of β and ϕ. Part of this is easy, because by inspection we only need to evaluate the lowest-order terms in

$$A_{kmk} = 0 + \cdots\,, \qquad B_{kmk} = \left(\frac{1}{2}\right) + \cdots\,, \tag{50}$$

because in this case it is adequate to use $\psi = 1 + \cdots$ because of the well-known orthogonality properties of sines and cosines. In contrast, the lowest order terms in A_{kmm} and B_{kmm} vanish identically and we must go to the second-order terms in ψ in order to get meaningful results. Thus, to the second order in the series (45c)

$$(A_{kmm}, B_{kmm}) = \left(\frac{3}{2}\right)\cdot(\varepsilon_{km})^2\cdot(\alpha_{kmm}, \beta_{kmm}) + \cdots\,, \tag{51a}$$

$$\alpha_{kmm} := \left(\frac{1}{[2\pi]}\right) \cdot \int_0^{2\pi} \sin(m_k\sigma) \cdot \cos(m_m\sigma) \times \cos([m_k - m_m] \cdot \sigma + \theta_k^0 - \theta_m^0)d\sigma \,, \tag{51b}$$

$$\beta_{kmm} := \left(\frac{1}{[2\pi]}\right) \cdot \int_0^{2\pi} \sin(m_k\sigma) \cdot \sin(m_m\sigma) \times \cos([m_k - m_m] \cdot \sigma + \theta_k^0 - \theta_m^0)d\sigma \,, \tag{51c}$$

which is where conceptualizing ends and labor begins. Our advice to the reader is to replace each sine and cosine by the sum or difference of two complex exponentials, as in de Moivre's theorem, and then multiply out the resulting 6 products as complex numbers. From this lengthy exercise in elementary complex algebra there results the (ultimately real) *identities*:

$$\alpha_{kmm} = -\left(\frac{1}{4}\right) \cdot \sin(\theta_k^0 - \theta_m^0) \,, \quad \beta_{kmm} = \left(\frac{1}{4}\right) \cdot \cos(\theta_k^0 - \theta_m^0) \,, \tag{51d}$$

from which we obtain the welcome simplification that $[(\alpha_{kmm})^2 + (\beta_{kmm})^2]^{1/2} \equiv 1/4$. Finally, by (46c)

$$\eta_{kmm} = \left(\frac{3}{2}\right) \cdot (\varepsilon_{km})^2 \cdot \left(\frac{1}{4}\right) + \cdots \,, \qquad \eta_{kmk} = \left(\frac{1}{2}\right) + \cdots \,, \tag{51e}$$

so that by (49a), to lowest order in μ,

$$\frac{1}{\beta} = \frac{\rho_k}{\rho_m} = \frac{\eta_{kmm}}{\eta_{kmk}} = \left(\frac{3}{2}\right) \cdot \frac{\rho_k \cdot \rho_m}{[(\rho_k)^2 + (\rho_m)^2]^{3/2}} \equiv \left(\frac{3}{2}\right) \cdot \frac{(1/\beta)}{[1 + (1/\beta)^2]^{3/2}} \,, \tag{51f}$$

which *requires* for self-consistency that, reminiscent of the *1766/1772 Titius/Bode 'law'*

$$1 + \left(\frac{1}{\beta}\right)^2 = \left(\frac{3}{2}\right)^{2/3} \,, \tag{51g}$$

i.e. that β have the *unique* particular value claimed in (30b)! Note that to lowest order this β is a *universal constant*, 'independent' of the gravitational constant constant G or the masses of the planets!

The proof of (30a) is analogous but simpler, noting that from (46d) and (51d)

$$\phi_{mk} = \Phi_{kmk} - \Phi_{kmm} = -(\theta_m^0 - \theta_k^0) \,, \tag{51h}$$

so that, bringing $(\theta_m^0 - \theta_k^0)$ to the left-hand side of (49b) and dividing by 2 we obtain the claimed result (30a). Equation (38) is a trivial consequence of (48b). also evaluation of the Jacobian with respect to (ρ, θ) instead of the components of makes $x(0)$ makes no difference to the claim that the Leray–Schauder index of the generating solution isolated by (51g) is unity. This completes the proof.

Now that the importance of the distal multiplier β for the distance beyond an inner planet for an outer planet in terms of the ratio of orbital radii has been derived in full rigor for the coplanar, concentric 3-body problem, it seems permissible to use a cruder physical model to consider the case of 4 or more bodies. For simplicity, keep the first body anchored at the origin and let the bodies have mean orbital radii ρ_i $(i = 1, 2, \ldots, n)$, where $\rho_1 \equiv 0$ by definition, and

$$0 = \rho_1 < \rho_2 < \cdots < \rho_i < \rho_{i+1} < \cdots < \rho_n \,, \tag{52}$$

and where each body (initially assumed decoupled from mutual interactions) is started with Keplerian circular velocity $v_i = \sqrt{G \cdot M}/\sqrt{\rho_i}$ where $M = m_1$ denotes the large central mass and where the smaller masses are denoted by $m_i = \mu_i \cdot M$ in terms of ratios $\mu_i \ll 1$ $(i = 2, 3, \ldots, n)$. Let $\Gamma_0 > 0$ denote initial total angular momentum, and $E_0 = |\mathcal{E}_0| = -\mathcal{E}_0 > 0$ denote the absolute value of the initial total energy. Also, define $\gamma_0 \equiv \Gamma_0/\sqrt{G} \cdot M^{3/2}$ and $\varepsilon_0 \equiv 2 \cdot E_0/\sqrt{G} \cdot M^2$, and it is easy to verify that (neglecting terms quadratic or higher in the μ_i) the conservation of angular momentum and conservation of energy (combined with the Virial theorem) are given by $\gamma = \gamma_0$ and $\varepsilon = \varepsilon_0$ where

$$\gamma = \sum_{i=2}^{n} \mu\sqrt{\rho_i}\,, \qquad \varepsilon = \sum_{i=2}^{n} \frac{\mu_i}{\rho_i}\,, \tag{53}$$

Theorem. *If the planetary orbits satisfy a Titius–Bode law of the form*

$$\rho_i = \beta^{i-2} \cdot \rho_2 \,, \quad (i = 2, 3, 4, \ldots, n) \,, \tag{54}$$

then the distal ratio $\beta = z^2$ *is the square of the unique positive root* $z > 0$ *of a polynomial of degree* $4 \cdot n - 8$ *in* z *whose coefficients are functions only of the constants* γ_0 *and* ε_0 *and the mass-ratios* μ_i *and which can be derived by elimination of* ρ_2 *between the expressions for* γ *and* ε *in* (53).

Proof. Insert (54) into (53) and compute $\gamma^2 \cdot \varepsilon$. I shall publish a general algorithm defining all of the coefficients explicitly elsewhere; however, it can be recovered easily by the reader after following the next example in the case $n = 3$, wherein the algebra is less difficult. □

Remark. To apply this result to the present solar system, simply replace the ρ_i in (53) by the actual mean distances R_i from observational astronomy and use the actual mass-ratios μ_i from astrophysics in order to evaluate γ_0 and ε_0 numerically. After finding β, the eliminated ρ_2 can be recovered by solving either $\gamma = \gamma_0$ or $\varepsilon = \varepsilon_0$, following which the remaining ρ_i are given by (54).

In the case $n = 3$, the polynomial $F(z)$ whose roots are sought has the form

$$F(z) = z^4 + \alpha_3 \cdot z^3 + \alpha_2 \cdot z^2 + \alpha_1 \cdot z^3 + \alpha_0 \,, \tag{55}$$

which, as in the proof of the preceding theorem, can be obtained simply by comparing coefficients in the multiplied-out version of

$$(\mu_1 \cdot z^2 + \mu_2) \cdot (\mu_1 + \mu_2 \cdot z)^2 - \alpha \cdot z^2 = 0 \,, \quad \alpha = \varepsilon_0 \cdot (\gamma_0)^2 \,. \tag{56}$$

Table 1.

m_i	R_i	ρ_i	Body	n	β	β_i
1,000,000			Sun	1		
0.16601	0.3871	0.2818	+Mercury	2		
2.447841	0.7233	0.50608	+Venus	3	1.86	1.868
3.003469	1.000	0.9088	+Earth	4	1.46	1.382
0.322714	1.5237	1.632	+Mars	5	1.47	1.5237
A				6		
954.60258	5.2030	5.26	+Jupiter	7	1.68	1.927
285.80765	9.5281	9.45	+Saturn	8	1.766	1.831
43.549846	19.1829	16.97	+Uranus	9	1.800	2.013
51.67161	30.0796	30.48	+Neptune	10	1.795	1.568
0.007541	49.0250	54.70	+Pluto	11	1.795	1.6298

The result is

$$\alpha_3 = 2 \cdot \left(\frac{\mu_1}{\mu_2}\right), \quad \alpha_2 = \left(\frac{\mu_1}{\mu_2}\right)^2 + \left(\frac{\mu_2}{\mu_1}\right) - \alpha[\mu_1(\mu_2)^2], \quad \alpha_1 = 2, \quad \alpha_0 = \frac{\mu_1}{\mu_2}. \qquad (57)$$

Using the data for Jupiter and Saturn, the reader may verify that $F = 0$ has only one positive root, whose square yields $\beta = 1.833$. With more effort, analogous results can be obtained for arbitrary n (see Table 1). Note that the value of b had converged to 1.795, in amazing agreement with the value obtained in (30b).

5. Conclusions

In this paper, we showed that in a planetary or satellite system, resonant perturbations will cause dynamical evolution of the orbital radii except for cases with highly specific mutual relationships. Simplifying the problem by reformulating it as a hierarchical sequence of (unrestricted) 3-body problems, in which gravitational interactions are ignored except those among the central body and the body of interest and the adjacent outer body, it is proved that the resonant perturbations from the outer body will destabilize the inner body (and vice versa) unless its mean orbital radius is a unique and specific multiple of β of that of the inner body. In this way a sequence of concentric orbits can each stabilize the adjacent inner orbit, until only the outermost orbit remains. Since the last orbit is already tied to the collection of inner orbits, by conservation of total angular momentum, the entire configuration is stabilized. Let $\mu = M_{\max}/M$ denote the ratio of the mass of the largest small body $M_{\max}$ to that of the large central body M; in our Solar system, μ is less than 10^{-3}. Expanding β in a power series in μ, the lowest-order terms for distal multiplier β and phase shift ϕ are found to start with the following universal constants (for $m = 1, 2, 3$)

$$\beta \equiv \beta + \cdots = \frac{1}{\sqrt{(3/2)^{(2/3)} - 1}} = 1.794980 + \cdots, \qquad \phi = m\left(\frac{\pi}{2}\right) + \cdots$$

which agrees with the observed distal ratios between Jupiter and the asteroids, Saturn and Jupiter, and Uranus and Saturn.

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