Gravitation Essay

APPENDIX 4

MACH'S PRINCIPLE IN A HYPERBOLIC COSMOLOGY

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ABSTRACT

Utilization of Mach's Principle in a manner precisely opposite that chosen by Jordan, Brans, and Dicke (but required by our analysis of all known solar system experiments pertaining to the measurement of the Eddington-Robertson parameter γ) yields quite rigorously the new result $G > [(|\omega| - 2)/3](c^3 t/M)$, $0 < t < +\infty$, $|\omega| > 2$, which is essentially the central idea of Milne's (hitherto neglected) cosmology. Also we find, approximately, $0 < (\dot{G}/G) \approx (3/|\omega|)qH \cong 3 \times 10^{-13}$, in terms of Sandage's 1974 values for the deceleration parameter $q \cong 0.05$ and inverse Hubble constant $H^{-1} \cong 2 \times 10^{10}$ years.

In standard General Relativity (GR) the gravitational field equations are (with c = 1)

(1) $$G_{\mu\nu} \stackrel{d}{=} R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}(R^{\lambda}{}_{\lambda}) = -8\pi G T_{\mu\nu}$$

where the Bianchi identities $G^{\mu}{}_{\nu;\,\mu} \equiv 0$ imply the accepted conservation laws $T^{\mu}{}_{\nu;\,\mu} = 0$. Following the formalism of Jordan, Brans, and Dicke (JBD) [cf. Jordan (1955); Dicke (1961)] we allow the Cavendish parameter G to be variable:

(2) $$G = 1/\phi,$$

(3) $$\Box^2\phi \stackrel{d}{=} \phi^{;\mu}{}_{;\mu} = -\frac{8\pi}{(2|\omega| - 3)}\, T^{\mu}{}_{\mu}$$

and generalize (1) to

(4) $$G_{\mu\nu} = -\frac{8\pi}{\phi}\,[T_{\mu\nu} + \hat{T}_{\mu\nu}]\ ,$$

(5) $$\hat{T}_{\mu\nu} = -\frac{|\omega|}{\phi}\,(\phi_{;\mu}\phi_{;\nu} - \tfrac{1}{2}\, g_{\mu\nu}\,[\phi^{;\lambda}\phi_{;\lambda}]) + $$
$$+ (\phi_{;\mu;\nu} - g_{\mu\nu}\,[\phi^{;\tau}{}_{;\tau}])\ ,$$

which is equivalent to

(6) $$R_{\mu\nu} = -\frac{8\pi}{\phi}\,[T_{\mu\nu} - \tfrac{1}{2}\,(\frac{2|\omega| - 2}{2|\omega| - 3})\, g_{\mu\nu}\,(T^{\lambda}{}_{\lambda})] + $$
$$+ \frac{|\omega|}{\phi^2}\,\phi_{;\mu}\,\phi_{;\nu} - \frac{1}{\phi}\,\phi_{;\mu;\nu}\ .$$

To recover the JBD theory, from (3)-(6), set

(7) $$\omega = -|\omega|$$

and assume that

(8) $$-3/2 < \omega < +\infty;$$

in JBD, one recovers GR as $\omega \to +\infty$. However, JBD leads to the conclusion that

(9) $$\dot{G}/G = -(\dot{\phi}/\phi) < 0\ ,$$

which we reject, because of abundant geophysical, astrophysical and cosmological evidence (presented elsewhere) that, with a 95% probability,

(10) $$\dot{G}/G > 0 .$$

Accordingly, in the present theory we assume that

(11) $$-\infty < \omega < -2 < 0 ,$$

i.e., we assume that

(12) $$2 < |\omega| < +\infty .$$

This parameter domain is disconnected from that of JBD, so the known results cannot be applied.

We intend to apply the theory (3), (6), (12) to the development of a hyperbolic cosmology. That is, we seek a homogeneous, isotropic expanding solution corresponding to a Robertson-Walker metric

(13) $$-ds^2 = d\tau^2 = dt^2 - R^2(t)\{(1 - kr^2)^{-1}dr^2 + r^2d\theta + r^2\sin^2\theta d\phi^2\}$$

with

(14) $$k = -1.$$

As shown by Weinberg (1972), the resulting field equations are

(15) $$\frac{3\ddot{R}}{R} = -8\pi\left(\frac{|\omega| - 2}{2|\omega| - 3}\right)\frac{\rho}{\phi} - \frac{\ddot{\phi}}{\phi} + |\omega|\left(\frac{\dot{\phi}}{\phi}\right)^2 ,$$

(16) $$(\dot{\phi}R^3)^{\cdot} \equiv \ddot{\phi}R^3 + 3R^2\dot{R}\dot{\phi} = -\left(\frac{8\pi}{2|\omega| - 3}\right)\rho R^3 ,$$

(17) $$\dot{\rho} = -\frac{3\dot{R}}{R}\rho ,$$

(18) $$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi}{3}\frac{\rho}{\phi} - \left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{\phi}}{\phi}\right) - \frac{|\omega|}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 ,$$

where t denotes cosmic time, $\cdot = d/dt$, ρ denotes density, and R denotes the cosmic "radius." Now (17) is equivalent to

(19) $$(\rho R^3)^{\cdot} \equiv \dot{\rho}R^3 + 3\rho R^2\dot{R} = 0 ,$$

whence we can define a constant M (essentially the rest-mass of the visible universe) by

(20) $$M = \frac{4}{3}\pi R^3 \rho \equiv \text{constant} \quad .$$

Now we may re-write (16) as

(21) $$(\dot{\phi}R^3)^{\bullet} \equiv \ddot{\phi}R^3 + 3R^2\dot{R}\dot{\phi} = -\phi_1 \quad ,$$

where

(22) $$\phi_1 \overset{d}{=} \frac{6M}{(2|\omega| - 3)} \quad ,$$

whence

(23) $$\frac{\ddot{\phi}}{\phi} + 3\left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{\phi}}{\phi}\right) + \left(\frac{\phi_1}{\phi}\right)\frac{1}{R^3} = 0 \; ,$$

(which, via

(24) $$\Box^2\phi \equiv -\left(\ddot{\phi} + 3\left(\frac{\dot{R}}{R}\right)\dot{\phi}\right)$$

is equivalent to (3)).

Integrating (21), we find that

(25) $$\dot{\phi}R^3 = -\phi_1 t \; ,$$

or, equivalently,

(26) $$\dot{\phi} = -\phi_1 \frac{t}{R^3} \quad .$$

Next, (20) and (26) yield

(27) $$\frac{\rho}{\phi} = -\left(\frac{2|\omega| - 3}{8\pi}\right)\left(\frac{\dot{\phi}}{\phi}\right)\frac{1}{t}$$

which, inserted in (15), together with (23), yields

(28) $$3\,\frac{\ddot{R}}{R} = \frac{(|\omega| - 2)}{t}\frac{\dot{\phi}}{\phi} + |\omega|\left(\frac{\dot{\phi}}{\phi}\right)^2 + 3\left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{\phi}}{\phi}\right) + \left(\frac{\phi_1}{\phi}\right)\frac{1}{R^3} =$$

$$= \left(\frac{\dot{\phi}}{\phi}\right)\left\{|\omega|\left(\frac{1}{t} + \frac{\dot{\phi}}{\phi}\right) + 3\left(\frac{\dot{R}}{R} - \frac{1}{t}\right)\right\}$$

or, recalling (9), and using the usual definitions

(29) $$H = \frac{\dot{R}}{R} \; , \; q = -\frac{\ddot{R}}{R}\left(\frac{1}{H^2}\right) \; ,$$

can be expressed in observable parameters as

(30) $$\boxed{qH^2 = \left(\frac{\dot{G}}{G}\right)\left\{\frac{|\omega|}{3}\left(\frac{1}{t} - \frac{\dot{G}}{G}\right) + \left(H - \frac{1}{t}\right)\right\}}.$$

Now insertion of (27) in (18) yields

(31) $$\left(\frac{\dot{R}}{R}\right)^2 = \frac{(-k)}{R^2} + \left|\frac{\dot{\phi}}{\phi}\right|\left\{\left(\frac{\dot{R}}{R} - \frac{1}{t}\right) + \frac{|\omega|}{2t} + \frac{|\omega|}{6}\left(\frac{1}{t} - \left|\frac{\dot{\phi}}{\phi}\right|\right)\right\}$$

which can be solved for $(\dot{R}/R)$ and integrated simultaneously with (26); i.e., by (14), the pair of ordinary differential equations

(32a) $$\boxed{\dot{\phi} = -\phi_1 \frac{t}{R^3},}$$

(32b) $$\boxed{\frac{\dot{R}}{R} = \tfrac{1}{2}\left[\left|\frac{\dot{\phi}}{\phi}\right| + \left\{\left(\frac{2|\omega| - 3}{3}\right)\left[\frac{4}{t} - \left|\frac{\dot{\phi}}{\phi}\right|\right]\left|\frac{\dot{\phi}}{\phi}\right| + \frac{4}{R^2}\right\}^{\frac{1}{2}}\right] > \frac{1}{R}}$$

gives the model we seek.

Note that, by (32b),

(33) $$\dot{R} > 1, \quad R > t \quad , \quad (0 < t < +\infty) \quad .$$

Now set

(34) $$\phi = \psi/t \; , \qquad R = t\Psi \; ,$$

and re-write the model (32a,b,) as

(35) $$\psi = \phi_1 t \int_t^{+\infty} \frac{1}{\Psi^3(\tau)\tau^2}\, d\tau \quad ,$$

(36) $$\Psi = \frac{1}{2t}\int_0^t \left(\frac{\phi_1}{\psi\Psi^2} + \left\{\left(\frac{2|\omega| - 3}{3}\right)\left[4\Psi - \frac{\phi_1}{\psi\Psi^2}\right]\frac{\phi_1}{\psi\Psi^2} + 4\right\}^{\frac{1}{2}}\right) d\tau \; .$$

Note that, corresponding to (33), equation (36) clearly implies that

(37) $$\Psi > 1 \quad , \qquad (0 < t < +\infty)$$

whence (35) yields, rigorously,

(38) $$\psi < \phi_1 \quad ,$$

i.e., we have proved that (recall c = 1)

$$(39) \qquad G > \left(\frac{|\omega| - 3}{3}\right) \frac{c^3 t}{M} \ , \quad (0 < t < + \infty) \ .$$

Now the central result of Milne's (1935) cosmology (which, as shown by McVittie (1949) can be developed in what Robertson and Noonan (1968) call an "expanding Minkowski metric) is that

$$(40) \qquad G \quad \alpha \quad \frac{c^3 t}{M} \ , \qquad (t \to + \infty) \ .$$

Milne's result has been ignored because it was not imbedded in accepted physics. Here, however, we have in (39) quite rigorously obtained Milne's central idea from a covariant gravitational field theory which is [cf. Misner, Thorne, and Wheeler (1973)] derivable from a Lagrangian variational principle.

As shown elsewhere, the usual (PPN) solar system measurements of G would not give $1/\phi$ but, rather,

$$(41) \qquad G = \left(\frac{2|\omega| - 4}{2|\omega| - 3}\right) \frac{1}{\phi} \ .$$

In conclusion, let us estimate $(\dot{G}/G)$ at the present epoch t. From standard Friedman model conceptions

$$(42) \qquad H = \frac{\alpha}{t} \ , \quad \alpha = O(1).$$

Then (30) can be re-written as

$$(43) \qquad qH = \left[\left(\frac{|\omega|}{3\alpha} + (1 - \frac{1}{\alpha}\right]\left(\frac{\dot{G}}{G}\right) - \frac{|\omega|}{3}\left(\frac{\dot{G}}{G}\right)^2\left(\frac{1}{H}\right) \ .$$

According to Sandage (1974), $H^{-1} \cong 2 \times 10^{10}$ yrs, $q \cong 0.05$. If, in (43), $(\dot{G}/G) = 3 \times 10^{-13}$, so that $(\dot{G}/G)^2 = 10^{-25}$, then the second term of (43) would be neglible in comparison to the first. Hence, for $|\omega| >> 2$, the result

$$(44) \qquad \boxed{0 < \frac{\dot{G}}{G} \cong \left(\frac{3}{|\omega|}\right) qH}$$

would be a reasonable approximate solution of (30); but now the cited numerical values, when inserted in (44), do give

$$0 < \dot{G}/G \cong 3 \times 10^{-13} \qquad (45)$$

which justifies the approximation made in (44). Note that (45) is far too small to be detectable by contemporary Earth-Moon laser measurements, which can at most detect effects $\cong 10^{-11}$.

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